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THE FIRST SIX BOOKS OF

EUCLID.

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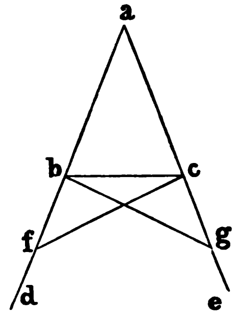
THE

FIRST SIX BOOKS

OF THE

ELEMENTS OF EUCLID.



WITH NUMEROUS EXERCISES.

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PREFACE.

THE present work, embracing the First Six Books of the *Elements of Euclid*, or in other words, the foundation of Plane Geometry, is intended as a companion volume to that issued with it on Practical Geometry; the one embracing the theory, the other the application.

The text adopted is that of the distinguished mathematician Dr. Simson. In very few instances has any change been made, and that only when it seemed to be positively necessary. Several attempts have been made, by altering the text and the arrangement, to reduce the demonstrations of Euclid to what may be considered a more popular form; but these have been attended with little success; and a text like that of Simson, clear, direct, and unencumbered, like absolute truth itself, will always be adopted by scholar and student.

It has been thought unnecessary to extend this volume beyond the first six books of the Elements, or to enlarge it by notes, as it will be followed by other works carrying the subject forward. A number of Exercises on each book have been given, the solutions of which will be published separately.

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THE ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I. A POINT is that which hath no parts, or which hath no magnitude.

II. A *line* is length without breadth.

III. The *extremities of a line* are points.

IV. A *straight line* is that which lies evenly between its extreme points.

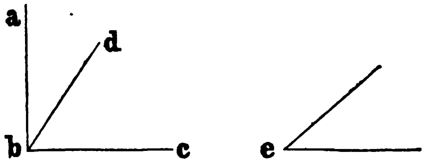
V. A *superficies* is that which hath only length and breadth.

VI. The *extremities of a superficies* are lines.

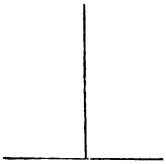
VII. A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII. A *plane angle* is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

IX. A *plane rectilineal angle* is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.



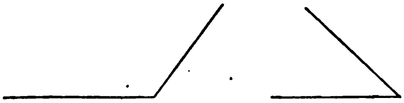
N.B.—When several angles are at one point b, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line. Thus the angle which is contained by the straight lines **ab**, **cb**, is named the angle **abc**, or **cba**; that which is contained by **ab**, **db**, is named the angle **abd**, or **dba**; and that which is contained by **db**, **cb**, is called the angle **dbc**, or **cbd**; but if there be only one angle at a point, it may be expressed by a letter placed at that point; as the angle at **e**.



X. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle*; and the straight line which stands on the other is called a *perpendicular* to it.

XI. An *obtuse angle* is that which is *greater* than a right angle.

XII. An *acute angle* is that which is *less* than a right angle.

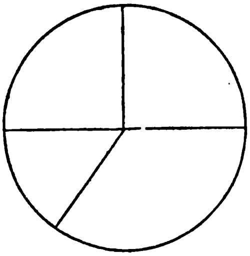


Obtuse. Acute.

XIII. A *term* or *boundary* is the extremity of any thing.

XIV. A *figure* is that which is enclosed by one or more boundaries.

XV. A *circle* is a plain figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI. And this point is called the *centre* of the circle.

XVII. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII. A *semicircle* is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX. A *segment* of a circle is the figure contained by a straight line, and the circumference it cuts off.

XX. *Rectilineal figures*, are those which are contained by straight lines.

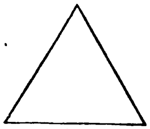
XXI. *Trilateral figures*, or triangles, by three straight lines.

XXII. *Quadrilateral*, by four straight lines.

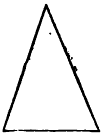
XXIII. *Multilateral figures*, or *polygons*, by more than four straight lines.

XXIV. Of three-sided figures, an *equilateral* triangle is that which has three equal sides.

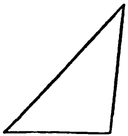
XXV. An *isosceles* triangle is that which has only two sides equal.



Equilateral.



Isosceles.

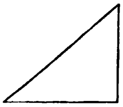


Scalene.

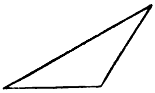
XXVI. A *scalene* triangle is that which has three unequal sides.

XXVII. A *right-angled* triangle is that which has a right angle.

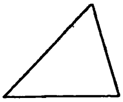
XXVIII. An *obtuse-angled* triangle is that which has an obtuse angle.



Right-angled.



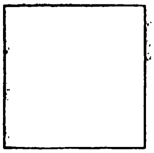
Obtuse-angled.



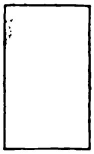
Acute-angled.

XXIX. An *acute-angled* triangle is that which has three acute angles.

XXX. Of four-sided figures, a *square* is that which has all its sides equal, and all its angles right angles.



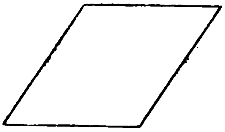
Square.



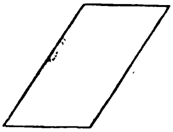
Oblong.

XXXI. An *oblong* is that which has all its angles right angles, but has not all its sides equal.

XXXII. A *rhombus* is that which has all its sides equal, but its angles are not right angles.



Rhombus.



Rhomboid.

XXXIII. A *rhomboid* is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

XXXIV. All other four-sided figures besides these are called *trapeziums*.

XXXV. *Parallel straight lines* are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

POSTULATES.

I. Let it be granted that a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS.

I. Things which are equal to the same thing are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals be added to unequals, the wholes are unequal.

V. If equals be taken from unequals, the remainders are unequal.

VI. Things which are double of the same, are equal to one another.

VII. Things which are halves of the same, are equal to one another.

VIII. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX. The whole is greater than its part.

X. Two straight lines cannot enclose a space.

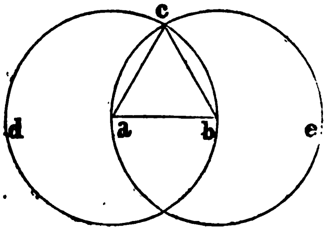
XI. All right angles are equal to one another.

XII. If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

Proposition I.—Problem.

To describe an equilateral triangle upon a given finite straight line.

Let **ab** be the given straight line; it is required to describe an equilateral triangle upon it.



From the centre **a**, at the distance **ab**, describe (3 *post.*) the circle **bcd**, and from the centre **b**, at the distance **ba**, describe the circle **ace**; and from the point **c**, in which the circles cut one another, draw the straight lines (1 *post.*) **ca**, **cb**, to the points **a**, **b**; **abc** shall be an equilateral triangle.

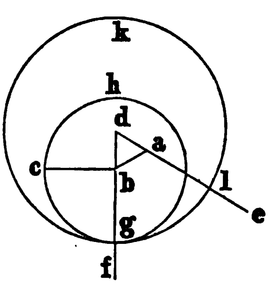
Because the point **a** is the centre of the circle **bcd**, **ac** is equal (15 *def.*) to

ab; and because the point **b** is the centre of the circle **ace**, **bc** is equal to **ba**: but it has been proved that **ca** is equal to **ab**; therefore **ca**, **cb**, are each of them equal to **ab**; but things which are equal to the same are equal to one another (1 *ax.*); therefore **ca** is equal to **cb**; wherefore **ca**, **ab**, **bc** are equal to one another; and the triangle **abc** is therefore equilateral, and it is described upon the given straight line **ab**. Which was to be done.

Proposition II.—Problem.

From a given point to draw a straight line equal to a given straight line.

Let **a** be the given point, and **bc** the given straight line; it is required to draw from the point **a** a straight line equal to **bc**.

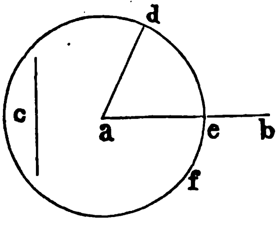


From the point **a** to **b** draw (1 *post.*) the straight line **ab**; and upon it describe (i. 1) the equilateral triangle **dab**, and produce (2 *post.*) the straight lines **da**, **db**, to **e** and **f**; from the centre **b**, at the distance **bc**, describe (3 *post.*) the circle **cgh**, and from the centre **d**, at the distance **dg**, describe the circle **gkl**. **al** shall be equal to **bc**.

Because the point **b** is the centre of the circle **cgh**, **bc** is equal (15 *def.*) to **bg**; and because **d** is the centre of the circle **gkl**, **dl** is equal to **dg**, and **da**, **db**, parts of them, are equal; therefore the remainder **al** is equal to the remainder (3 *ax.*) **bg**; but it has been shewn that **bc** is equal to **bg**; wherefore **al** and **bc** are each of them equal to **bg**; and things that are equal to the same are equal to one another; therefore the straight line **al** is equal to **bc**. Wherefore from the given point **a** a straight line **al** has been drawn equal to the given straight line **bc**. Which was to be done.

Proposition III.—Problem.

From the greater of two given straight lines to cut off a part equal to the less.

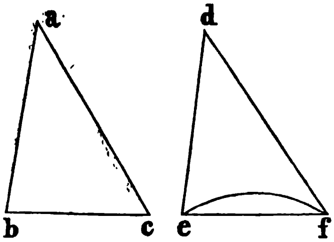


Let **ab** and **c** be the two given straight lines, whereof **ab** is the greater. It is required to cut off from **ab**, the greater, a part equal to **c**, the less.

From the point **a** draw (i. 2) the straight line **ad** equal to **c**; and from the centre **a**, and at the distance **ad**, describe (3 *post.*) the circle **def**; **ae** shall be equal to **c**. Because **a** is the centre of the circle **def**, **ae** is equal to **ad**; but the straight line **c** is likewise equal to **ad**; whence **ae** and **c** are each of them equal to **ad**; therefore the straight line **a e** is equal (1 *ax.*) to **c**, and from **a b**, the greater of two straight lines, a part **a e** has been cut off equal to **c** the less. Which was to be done.

PROPOSITION IV.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases, or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*



LET **a b c**, **d e f** be two triangles, which have the two sides **a b**, **a c** equal to the two sides **d e**, **d f**, each to each, viz. **a b** to **d e**, and **a c** to **d f**; and the angle **b a c** equal to the angle **e d f**, the base **b c** shall be equal to the base **e f**; and the triangle **a b c** to the triangle **d e f**; and the other angles to which the equal sides are opposite, shall be equal, each to each, viz. the angle **a b c** to the angle **d e f**, and the angle **a c b** to **d f e**.

For, if the triangle **a b c** be applied to **d e f**, so that the point **a** may be on **d**, and the straight line **a b** upon **d e**; the point **b** shall coincide with the point **e**, because **a b** is equal to **d e**; and **a b** coinciding with **d e**, **a c** shall coincide with **d f**, because the angle **b a c** is equal to the angle **e d f**; wherefore also the point **c** shall coincide with the point **f**, because the straight line **a c** is equal to **d f**: but the point **b** coincides with the point **e**; wherefore the base **b c** shall coincide with the base **e f**, because the point **b** coinciding with **e**, and **c** with **f**, if the base **b c** does not coincide with the base **e f**, two straight lines would enclose a space, which is impossible (10 *ax.*). Therefore the base **b c** shall coincide with the base **e f**, and be equal to it. Wherefore the whole triangle **a b c** shall coincide with the whole triangle **d e f**, and be equal to it; and the other angles of the one shall coincide with the remaining angles of the other, and be equal to them, viz. the angle **a b c** to the angle **d e f**, and the angle **a c b** to **d f e**. Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases shall likewise be equal, and the triangles be equal, and their other angles to which the equal sides are opposite shall be equal, each to each. Which was to be demonstrated.

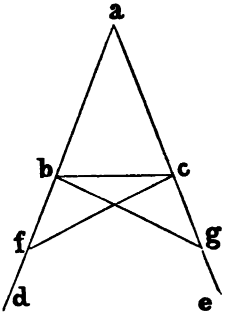
PROPOSITION V.—THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

LET **a b c** be an isosceles triangle, of which the side **a b** is equal to **a c**, and let the straight lines **a b**, **a c** be produced to **d** and **e**, the angle **a b c** shall be equal to the angle **a c b**, and the angle **c b d** to the angle **b c e**.

In **bd** take any point **f**, and from **ae** the greater, cut off **ag** equal (i. 3) to **af**, the less, and join **fc**, **gb**.

Because **af** is equal to **ag**, and **ab** to **ac**, the two sides **fa**, **ac** are equal to the two **ga**, **ab**, each to each; and they contain the angle **fag** common to the two triangles **afc**, **agb**; therefore the base **fc** is equal (i. 4) to the base **gb**, and the triangle **afc** to the triangle **agb**; and the remaining angles of the one are equal (i. 4) to the remaining angles of the other, each to each, to which the equal sides are opposite; viz. the angle **acf** to the angle **abg**, and the angle **afc** to the angle **agb**: and because the whole **af** is equal to the whole **ag**, of which the parts **ab**, **ac**, are equal; the remainder **bf** shall be equal (3 *ax.*) to the remainder **cg**; and **fc** was proved to be equal to **gb**; therefore the two sides **bf**, **fc** are equal to the two **cg**, **gb**, each to each; and the angle **bfc** is equal to the angle **cgb**, and the base **bc** is common to the two triangles **bfc**, **cgb**; wherefore the triangles are equal (i. 4), and their remaining angles, each to each, to which the equal sides are opposite; therefore the angle **fbc** is equal to the angle **gcb**, and the angle **bcf** to the angle **cbg**: and since it has been demonstrated, that the whole angle **abg** is equal to the whole **acf**, the parts of which, the angles **cbg**, **bcf** are also equal; the remaining angle **abc** is therefore equal to the remaining angle **acb**, which are the angles at the base of the triangle **abc**: and it has also been proved that the angle **fbc** is equal to the angle **gcb**, which are the angles upon the other side of the base. Therefore the angles at the base, &c. Q. E. D.



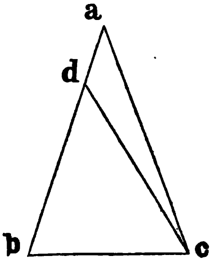
COROLLARY. Hence every equilateral triangle is also equiangular.

PROPOSITION VI.—THEOREM.

If two angles of a triangle be equal to one another, the sides which subtend, or are opposite to, the equal angles, shall also be equal to one another.

LET **abc** be a triangle having the angle **abc** equal to the angle **acb**; the side **ab** is also equal to the side **ac**.

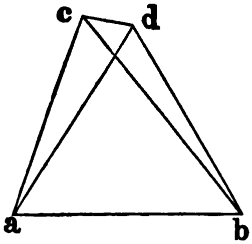
For, if **ab** be not equal to **ac**, one of them is greater than the other: let **ab** be the greater; and from it cut (i. 3) off **db** equal to **ac**, the less, and join **dc**; therefore, because in the triangles **dbc**, **acb**, **db** is equal to **ac**, and **bc** common to both, the two sides, **db**, **bc** are equal to the two **ac**, **cb**, each to each; and the angle **dbc** is equal to the angle **acb**; therefore the base **dc** is equal to the base **ab**, and the triangle **dbc** is equal to the triangle (i. 4) **acb**, the less to the greater; which is absurd. Therefore **ab** is not unequal to **ac**, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D.



COR. Hence every equiangular triangle is also equilateral.

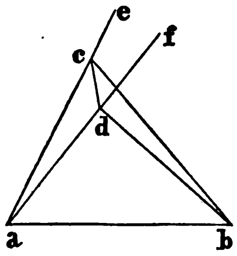
PROPOSITION VII.—THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity.



IF it be possible, let there be two triangles **a c b**, **a d b**, upon the same base **a b**, and upon the same side of it, which have their sides **c a**, **d a** terminated in the extremity **a** of the base equal to one another, and likewise their sides **c b**, **d b**, that are terminated in **b**.

Join **c d**; then, in the case in which the vertex of each of the triangles is without the other triangle, because **a c** is equal to **a d**, the angle **a c d** is equal (i. 5) to the angle **a d c**: but the angle **a c d** is greater than the angle **b c d**; therefore the angle **a d c** is greater also than **b c d**; much greater then is the angle **b d c** than the angle **b c d**. Again, because **c b** is equal to **d b**, the angle **b d c** is equal (i. 5) to the angle **b c d**; but it has been demonstrated to be greater than it; which is impossible.



But if one of the vertices, as **d**, be within the other triangle **a c b**; produce **a c**, **a d** to **e**, **f**; therefore, because **a c** is equal to **a d** in the triangle **a c d**, the angles **e c d**, **f d c** upon the other side of the base **c d** are equal (i. 5) to one another: but the angle **e c d** is greater than the angle **b c d**: wherefore the angle **f d c** is likewise greater than **b c d**; much greater then is the angle **b d c** than the angle **b c d**. Again, because **c b** is equal to **d b**, the angle **b d c** is equal (i. 5) to the angle **b c d**; but **b d c** has been proved to be greater than the same **b c d**; which is impossible.

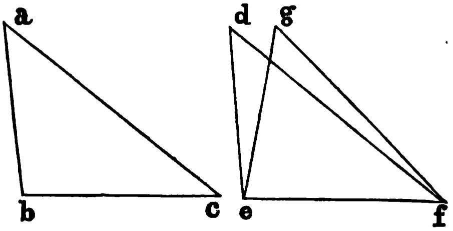
The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

Therefore, upon the same base, and on the same side of it, &c. Q. E. D.

PROPOSITION VIII.—THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.

LET **a b c**, **d e f** be two triangles, having the two sides **a b**, **a c**, equal to the two sides **de**, **df**, each to each, viz. **ab** to **de**, and **ac** to **df**; and also the base **bc** equal to the base **ef**. The angle **bac** is equal to the angle **edf**.



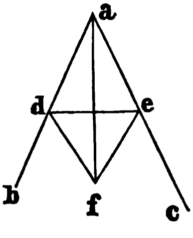
For, if the triangle **abc** be applied to **def**, so that the point **b** be on **e**, and the straight line **bc** upon **ef**; the point **c** shall also coincide with the point **f**, because **bc** is equal to **ef**. Therefore **bc** coinciding with **ef**; **ba** and **ac** shall coincide with **ed** and **df**; for, if the base **bc** coincides with the base **ef**, but the sides **ba**, **ca** do not coincide with the sides **ed**, **fd**, but have a different situation, as **eg**, **fg**, then upon the same base **ef**, and upon the same side of it, there can be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise their sides terminated in the other extremity; but this is impossible (i. 7); therefore, if the base **bc** coincides with the base **ef**, the sides **ba**, **ac**, cannot but coincide with the sides **ed**, **df**; wherefore likewise the angle **bac** coincides with the angle **edf**, and is equal (8*ax.*) to it. Therefore if two triangles, &c. Q.E.D.

Proposition IX.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let **bac** be the given rectilineal angle, it is required to bisect it.

Take any point **d** in **ab**, and from **ac** cut (i. 3) off **ae** equal to **ad**; join **de**, and upon it describe (i. 1) an equilateral triangle **def**; then join **af**; the straight line **af** bisects the angle **bac**.

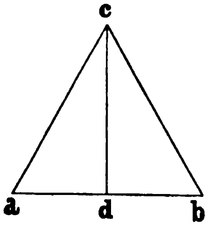


Because **ad** is equal to **ae**, and **af** is common to the two triangles **daf**, **eaf**; the two sides **da**, **af**, are equal to the two sides **ea**, **af**, each to each; and the base **df** is equal to the base **ef**; therefore the angle **daf** is equal (i. 8) to the angle **eaf**; wherefore the given rectilineal angle **bac** is bisected by the straight line **af**. Which was to be done.

Proposition X.—Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let **ab** be the given straight line, it is required to divide it into two equal parts.



Describe (i. 1) upon **a b** an equilateral triangle **a b c**, and bisect (i. 9) the angle **a c b** by the straight line **c d**. **a b** is cut into two equal parts in the point **d**.

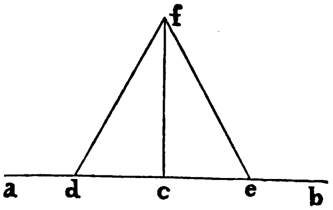
Because **a c** is equal to **c b**, and **c d** common to the two triangles **a c d**, **b c d**; the two sides **a c**, **c d** are equal to **b c**, **c d**, each to each; and the angle **a c d** is equal to the angle **b c d**; therefore the base **a d** is equal to the base (i. 4) **d b**, and the straight line **a b** is divided into two equal parts in the point **d**. Which was to be done.

Proposition XI.—Problem.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let **a b** be a given straight line, and **c** a point given in it; it is required to draw a straight line from the point **c** at right angles to **a b**.

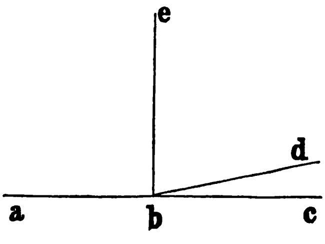
Take any point **d** in **a c**, and make (i. 3) **c e** equal to **c d**, and upon **d e** describe (i. 1) the equilateral triangle **d f e**, and join **f c**, the straight line **f c** drawn from the given point **c** is at right angles to the given straight line **a b**.



Because **d c** is equal to **c e**, and **f c** common to the two triangles **d c f**, **e c f**; the two sides **d c**, **c f**, are equal to the two **e c**, **c f**, each to each; and the base **d f** is equal to the base **e f**; therefore the angle **d c f** is equal (i. 8) to the angle **e c f**; and they are adjacent angles. But, when the adjacent angles which one straight line makes with another straight line are equal to one another, each of them is called a right angle (10 *def.*); therefore each of the angles **d c f**, **e c f**, is a right angle. Wherefore, from the given point **c**, in the given straight line **a b**, **f c** has been drawn at right angles to **a b**. Which was to be done.

Cor. By help of this problem, it may be demonstrated, that two straight lines cannot have a common segment.

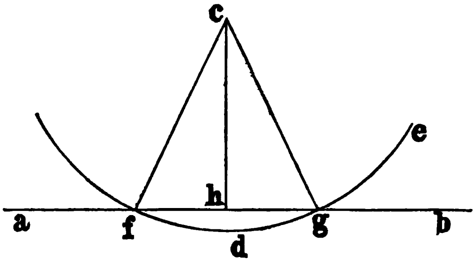
If it be possible, let the two straight lines **a b c**, **a b d** have the segment **a b** common to both of them. From the point **b** draw **b e** at right angles to **a b**; and because **a b c** is a straight line, the angle **c b e** is equal (10 *def.*) to the angle **e b a**; in the same manner, because **a b d** is a straight line, the angle **d b e** is equal to the angle **e b a**; wherefore the angle **d b e** is equal to the angle **c b e**, the less to the greater, which is impossible; therefore two straight lines cannot have a common segment.



PROPOSITION XII.—PROBLEM.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

LET **a b** be the given straight line, which may be produced to any length both ways, and let **c** be a point without it. It is required to draw a straight line perpendicular to **a b** from the point **c**.



Take any point **d** upon the other side of **a b**, and from the centre **c**, at the distance **c d**, describe (3 *post.*) the circle **e g f** meeting **a b** in **f**, **g**; and bisect (i. 10) **f g** in **h**, and join **c f**, **c h**, **c g**; the straight line **c h**, drawn from the given point **c**, is perpendicular to the given straight line **a b**.

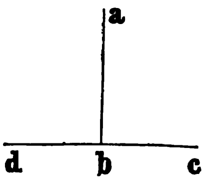
Because **f h** is equal to **h g**, and **h c** common to the two triangles **f h c**, **g h c**, the two sides **f h**, **h c** are equal to the two **g h**, **h c**, each to each; and the base **c f** is equal (15 *def.*) to the base **c g**; therefore the angle **c h f** is equal (i. 8) to the angle **c h g**; and they are adjacent angles; but when (10 *def.*) a straight line standing on another straight line makes the adjacent angles equal to one another, each of them is a right angle; and the straight line which stands upon the other is called a perpendicular to it; therefore from the given point **c** a perpendicular **c h** has been drawn to the given straight line **a b**. Which was to be done.

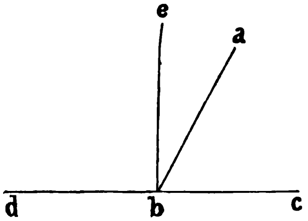
PROPOSITION XIII.—THEOREM.

The angles which one straight line makes with another upon one side of it are either two right angles, or are together equal to two right angles.

LET the straight line **a b** make with **c d**, upon one side of it, the angles **c b a**, **a b d**: these are either two right angles, or are together equal to two right angles.

For if the angle **c b a** be equal to **a b d**, each of them is a right angle



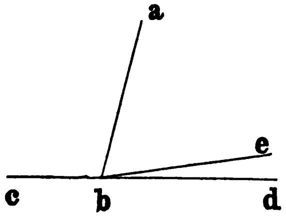


(10 *def.*); but if not, from the point **b** draw **b e** at right angles (i. 11) to **c d**; therefore the angles **c b e**, **e b d** are two right angles (10 *def.*); and because **c b e** is equal to the two angles **c b a**, **a b e** together, add the angle **e b d**

to each of these equals; therefore the angles **c b e**, **e b d** are equal (2 *ax.*) to the three angles **c b a**, **a b e**, **e b d**. Again, because the angle **d b a** is equal to the two angles **d b e**, **e b a**, add to these equals the angle **a b c**, therefore the angles **d b a**, **a b c** are equal to the three angles **d b e**, **e b a**, **a b c**: but the angles **c b e**, **e b d** have been demonstrated to be equal to the same three angles; and things that are equal to the same are equal (1 *ax.*) to one another; therefore the angles **c b e**, **e b d** are equal to the angles **d b a**, **a b c**; but **c b e**, **e b d** are two right angles; therefore **d b a**, **a b c** are together equal to two right angles. Wherefore, the angles which one straight line, &c. Q. E. D.

PROPOSITION XIV.—THEOREM.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.



AT the point **b** in the straight line **a b**, let the two straight lines **b c**, **b d** upon the opposite sides of **a b**, make the adjacent angles **a b c**, **a b d** equal together to two right angles, **b d** is in the same straight line with **c b**.

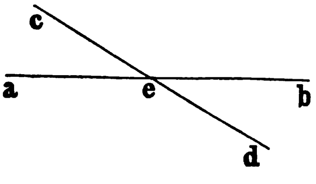
For, if **b d** be not in the same straight line with **c b**, let **b e** be in the same straight line with it; therefore, because the straight line **a b** makes angles with the straight line **c b e**, upon one side of it, the angles **a b c**, **a b e** are together equal (i. 13) to two right angles; but the angles **a b c**, **a b d** are likewise together equal to two right angles; therefore the angles **c b a**, **a b e** are equal to the angles **c b a**, **a b d**: take away the common angle **c b a**, the remaining angle **a b e** is equal (3 *ax.*) to the remaining angle **a b d**, the less to the greater, which is impossible; therefore **b e** is not in the same straight line with **b c**. And in like manner it may be demonstrated, that no other can be in the same straight line with it but **b d**, which therefore is in the same straight line with **c b**. Wherefore, if at a point, &c. Q. E. D.

PROPOSITION XV.—THEOREM.

If two straight lines cut one another, the vertical, or opposite, *angles shall be equal.*

LET the two straight lines **a b**, **c d** cut one another in the point **e**; the angle **a e c** shall be equal to the angle **d e b**, and **c e b** to **a e d**.

Because the straight line **a e** makes with **c d** the angles **c e a**, **a e d**, these angles are together equal (i. 13) to two right angles. Again, because the straight line **d e** makes with **a b** the angles **a e d**, **d e b**, these also are together equal (i. 13) to two right angles; and **c e a**, **a e d**, have been demonstrated to be equal to two right angles; wherefore the angles **c e a**, **a e d**,



are equal to the angles **a e d**, **d e b**. Take away the common angle **a e d**, and the remaining angle **c e a** is equal (3 *ax.*) to the remaining angle **d e b**. In the same manner it can be demonstrated, that the angles **c e b**, **a e d** are equal. Therefore, if two straight lines, &c. Q. E. D.

COR. 1. From this it is manifest, that, if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

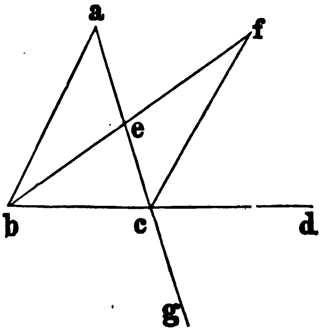
COR. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROPOSITION XVI.—THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

LET **a b c** be a triangle, and let its side **b c** be produced to **d**, the exterior angle **a c d** is greater than either of the interior opposite angles **c b a**, **b a c**.

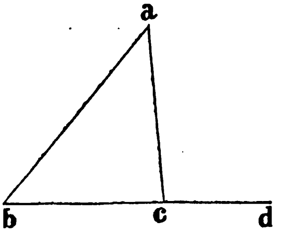
Bisect (i. 10) **a c** in **e**, join **b e** and produce it to **f**, and make **e f** equal to **b e**; join also **f c**, and produce **a c** to **g**.



Because **a e** is equal to **e c**, and **b e** to **e f**; **a e**, **e b** are equal to **c e**, **e f**, each to each; and the angle **a e b** is equal (i. 15) to the angle **c e f**, because they are opposite vertical angles; therefore the base **a b** is equal (i. 4) to the base **c f**, and the triangle **a e b** to the triangle **c e f**, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle **b a e** is equal to the angle **e c f**; but the angle **e c d** is greater than the angle **e c f**; therefore the angle **a c d** is greater than **b a e**. In the same manner, if the side **b c** be bisected, it may be demonstrated that the angle **b c g**, that is, (i. 15) the angle **a c d**, is greater than the angle **a b c**. Therefore, if one side, &c. Q. E. D.

PROPOSITION XVII.—THEOREM.

Any two angles of a triangle are together less than two right angles.

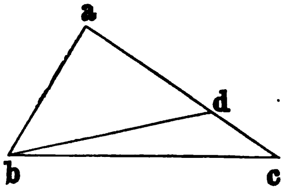


LET **a b c** be any triangle; any two of its angles together are less than two right angles.

Produce **b c** to **d**; and because **a c d** is the exterior angle of the triangle **a b c**, **a c d** is greater (i. 16) than the interior and opposite angle **a b c**; to each of these add the angle **a c b**; therefore the angles **a c d**, **a c b** are greater than the angles **a b c**, **a c b**; but **a c d**, **a c b** are together equal (i. 13) to two right angles; therefore the angles **a b c**, **b c a** are less than two right angles. In like manner it may be demonstrated, that **bac**, **acb**, as also **cab**, **abc**, are less than two right angles. Therefore any two angles, &c. Q. E. D.

Proposition XVIII.—Theorem.

The greater side of every triangle is opposite to the greater angle.



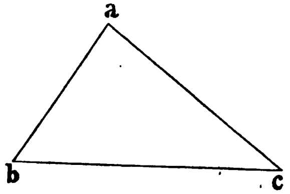
Let **abc** be a triangle, of which the side **ac** is greater than the side **ab**; the angle **abc** is also greater than the angle **bca**.

Because **ac** is greater than **ab**, make (i. 3) **ad** equal to **ab**, and join **bd**; and because **adb** is the exterior angle of the triangle **bdc**, it is greater (i. 16) than the interior and opposite angle **dcb**; but **adb** is equal (i. 5) to **abd**, because the side **ab** is equal to the side **ad**; therefore the angle **abd** is likewise greater than the angle **acb**. Much greater then is the angle **abc** than **acb**. Therefore the greater side, &c. Q. E. D.

Proposition XIX.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Let **abc** be a triangle, of which the angle **abc** is greater than the angle **bca**; the side **ac** is likewise greater than the side **ab**.



For, if it be not greater, **ac** must either be equal to **ab**, or less than it; it is not equal, because then the angle **abc** would be equal (i. 5) to the angle **acb**; but it is not; therefore **ac** is not equal to **ab**; neither is it less; because then the angle **abc** would be less than the angle **acb** (i. 18); but it is not; therefore the side **ac** is not less than **ab**; and it has been shewn that it is not equal to **ab**; therefore **ac** is greater than **ab**. Wherefore the greater angle, &c. Q. E. D.

Proposition XX.—Theorem.

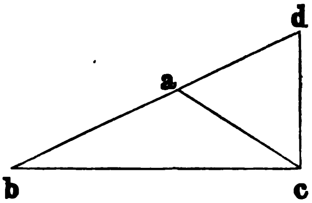
Any two sides of a triangle are together greater than the third side.

Let **abc** be a triangle; any two sides of it together are greater than the third side, viz. the sides **ba**, **ac** greater than the side **bc**; and **ab**, **bc** greater than **ac**; and **bc**, **ca** greater than **ab**.

Produce **ba** to the point **d**, and make (i. 3) **ad** equal to **ac**; and join **dc**.

Because **da** is equal to **ac**, the angle **adc** is likewise equal to **acd** (i. 5); but the angle **bcd** is greater than the angle **acd**; therefore the angle

bcd is greater than the angle **adc**; and because the angle **bcd** of the triangle **dcb** is greater than its angle **bdc**, and that the greater (i. 19) side is opposite to the greater angle, therefore the side **db** is greater than the side **bc**; but **db** is equal to **ba** and **ac**; therefore the sides **ba**, **ac** are greater than **bc**. In the same manner it may be demonstrated, that the sides **ab**, **bc**, are greater than **ca**, and **bc**, **ca** greater than **ab**. Therefore any two sides, &c. Q. E. D.

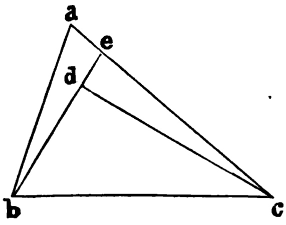


Proposition XXI.—Theorem.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let the two straight lines **bd**, **cd** be drawn from **b**, **c**, the ends of the side **bc** of the triangle **abc**, to the point **d** within it; **bd** and **dc** are less than the other two sides **ba**, **ac** of the triangle, but contain an angle **bdc** greater than the angle **bac**.

Produce **bd** to **e**; and because two sides of a triangle are greater than the third side, the two sides **ba**, **ae**, of the triangle **abe** are greater than **be**. To each of these add **ec**; therefore the sides **ba**, **ac** are greater than **be**, **ec**. Again, because the two sides **ce**, **ed** of the triangle **ced** are greater than **cd**, add **db** to each of these; therefore the sides **ce**, **eb** are greater than **cd**, **db**; but it has been shewn that **ba**, **ac**, are greater than **be**, **ec**, much greater then are **ba**, **ac** than **bd**, **dc**.



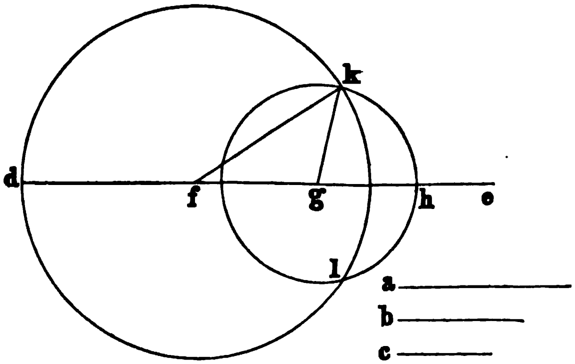
Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle **bdc** of the triangle **cde** is greater than **ced**; for the same reason, the exterior angle **ceb** of the triangle **abe** is greater than **bac**; and it has been demonstrated that the angle **bdc** is greater than the angle **ceb**; much greater then is the angle **bdc** than the angle **bac**. Therefore, if from the ends of, &c. Q. E. D.

Proposition XXII.—Problem.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third (i. 20.).

Let **a**, **b**, **c** be the three given straight lines, of which any two whatever are greater than the third, viz. **a** and **b** greater than **c**; **a** and **c** greater than **b**; and **b** and **c** than **a**. It is required to make a triangle of which the sides shall be equal to **a**, **b**, **c**, each to each.

Take a straight line de terminated at the point d, but unlimited towards e, and make (i. 3) df equal to a, fg to b, and gh equal to c; and from the centre f, at the distance fd, describe (3 *post.*) the circle dkl;

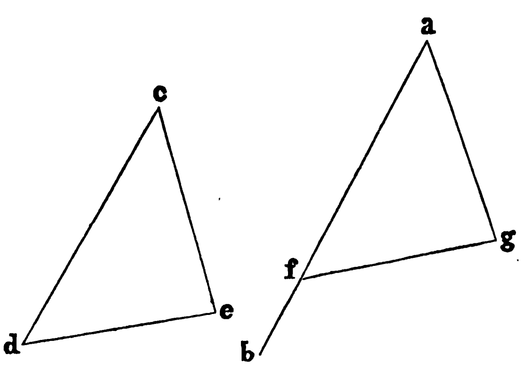


and from the centre g, at the distance gh, describe (3 *post.*) another circle hlk; and join kf, kg; the triangle kfg has its sides equal to the three straight lines a, b, c.

Because the point f is the centre of the circle dkl, fd is equal (15 *def.*) to fk; but fd is equal to the straight line a; therefore fk is equal to a: again, because g is the centre of the circle lkh, gh is equal (15 *def.*) to gk; but gh is equal to c; therefore also gk is equal to c; and fg is equal to b; therefore the three straight lines kf, fg, gk, are equal to the three a, b, c: and therefore the triangle kfg has its three sides kf, fg, gk, equal to the three given straight lines a, b, c. Which was to be done.

PROPOSITION XXIII.—PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.



LET ab be the given straight line, and a the given point in it, and dce the given rectilineal angle; it is required to make an angle at the given point a in the given straight line ab, that shall be equal to the given rectilineal angle dce.

Take in cd, ce any points d, e, and join de; and make (i. 22) the triangle afg, the sides of which shall be equal to the three straight lines cd, de, ec, so that cd be equal to af, ce to ag, and de to fg; and because dc, ce are equal to **fa**, **ag**, each to each, and the base **de** to the base **fg**; the angle **dce** is equal (i. 8) to the angle **fag**. Therefore, at the given point **a** in the given straight line **ab**, the angle **fag** is made equal to the given rectilineal angle **dce**. Which was to be done.

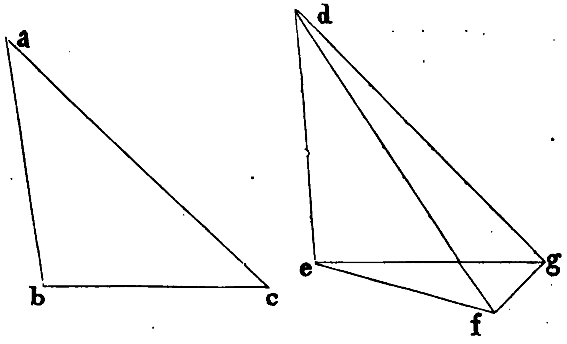
Proposition XXIV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let **abc**, **def** be two triangles, which have the two sides **ab**, **ac**, equal to the two **de**, **df**, each to each, viz. **ab** equal to **de**, and **ac** to **df**; but the angle **bac** greater than the angle **edf**; the base **bc** is also greater than the base **ef**.

Of the two sides **de**, **df**, let **de** be the side which is not greater than the other, and at the point **d**, in the straight line **de**, make (i. 23) the angle **edg** equal to the angle **bac**; and make **dg** equal (i. 3) to **ac** or **df**, and join **eg**, **gf**.

Because **ab** is equal to **de**, and **ac** to **dg**, the two sides **ba**, **ac** are equal to the two, **ed**, **dg**, each to each, and the angle **bac** is equal to the angle **edg**; therefore the base **bc** is equal (i. 4) to the base **eg**; and because **dg** is equal to **df**, the angle **dfg** is equal (i. 5) to the angle **dgf**; but the angle **dgf** is greater than the angle **egf**; therefore the angle **dfg** is greater than **egf**; and much greater is the angle **efg** than the angle **egf**; and because the angle **efg** of the triangle **efg** is greater than its angle **egf**, and that the greater (i. 19) side is opposite to the greater angle; the side **eg** is therefore greater than the side **ef**; but **eg** is equal to **bc**; and therefore also **bc** is greater than **ef**. Therefore, if two triangles, &c. Q. E. D.

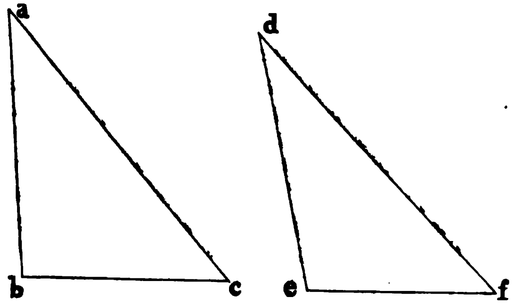


Proposition XXV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

Let **abc**, **def** be two triangles which have the two sides **ab**, **ac** equal to the two sides **de**, **df**, each to each, viz. **ab** equal to **de**, and **ac** to **df**; but the base **bc** greater than the base **ef**; the angle **bac** is likewise greater than the angle **edf**.

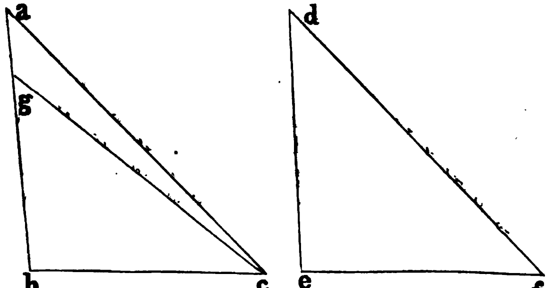
For, if it be not greater, it must either be equal to it, or less; but the angle **bac** is not equal to the angle **edf**; because then the base **bc** would be equal (i. 4) to **ef**; but it is not; therefore the angle **bac** is not equal to the angle **edf**; neither is it less: because then the base **bc** would be less (i. 24) than the base **ef**; but it is not; therefore the angle **bac** is not less than the angle **edf**; and it was shewn that it is not equal to it; therefore the angle **bac** is greater than the angle **edf**. Wherefore, if two triangles, &c. Q. E. D.



PROPOSITION XXVI.—THEOREM.

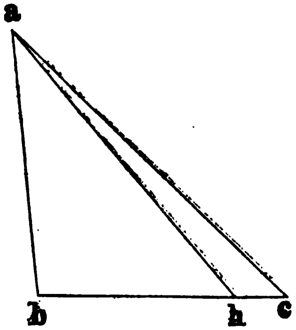
If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each, and also the third angle of the one to the third angle of the other.

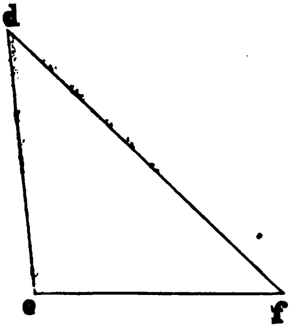
LET **abc**, **def** be two triangles which have the angles **abc**, **bca** equal to the angles **def**, **efd**, viz. **abc** to **def**, and **bca** to **efd**; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles: viz. **bc** to **ef**; the other sides shall be equal each to each, viz. **ab** to **de**, and **ac** to **df**, and the third angle **bac** to the third angle **edf**.



For, if **ab** be not equal to **de**, one of them must be the greater. Let **ab** be the greater of the two, and make **bg** equal to **de**, and join **gc**; therefore, because **bg** is equal to **de**, and **bc** to **ef**, the two sides **gb**, **bc** are equal to the two **de**, **ef**, each to each; and the angle **gbc** is equal to the angle **def**; therefore the base **gc** is equal (i. 4) to the base **df**, and the triangle **gbc** to the triangle **def**, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle **gcb** is equal to the angle **dfe**; but **dfe** is, by the hypothesis, equal to the angle **bca**; wherefore also the angle **bcg** is equal to the angle **bca**, the less to the greater, which is impossible; therefore **ab** is not unequal to **de**, that is, it is equal to it; and **bc** is equal to **ef**; therefore the two **ab**, **bc** are equal to the two **de**, **ef**, each to each; and the angle **abc** is equal to the angle **def**; the base therefore **ac** is equal (i. 4) to the base **df**, and the third angle **bac** to the third angle **edf**.

Next, let the sides which are opposite to equal angles in each triangle be equal to one another, viz. **ab** to **de**; likewise in this case, the other sides shall be equal, **ac** to **df**, and **bc** to **ef**; and also the third angle **bac** to the third **edf**.





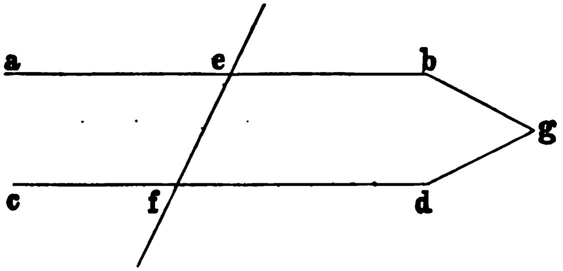
For, if **bc** be not equal to **ef**, let **bc** be the greater of them, and make **bh** equal to **ef**, and join **ah**; and because **bh** is equal to **ef**, and **ab** to **de**; the two **ab**, **bh** are equal to the two **de**, **ef**, each to each; and they contain equal angles; therefore the base **ah** is equal to the base **df**, and the triangle **abh** to the triangle **def**, and the other angles shall be equal, each to each, to which the equal sides are opposite; therefore the angle **bha** is equal to the angle **efd**; but **efd** is equal to the angle **bca**; therefore also the angle **bha** is equal to the angle **bca**, that is, the exterior angle **bha** of the triangle **ahc** is equal to its interior and opposite angle **bca**; which is impossible (i. 16); wherefore **bc** is not unequal to **ef**, that is, it is equal to it; and **ab** is equal to **de**; therefore the two, **ab**, **bc** are equal to the two **de**, **ef**, each to each; and they contain equal angles; wherefore the base **ac** is equal to the base **df**, and the third angle **bac** to the third angle **edf**. Therefore, if two triangles, &c. Q. E. D.

PROPOSITION XXVII.—THEOREM.

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

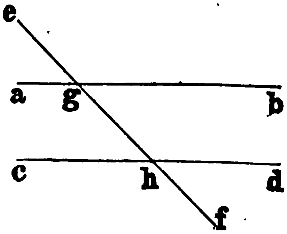
LET the straight line **ef**, which falls upon the two straight lines **ab**, **cd**, make the alternate angles **aef**, **efd** equal to one another; **ab** is parallel to **cd**.

For, if it be not parallel, **a b** and **c d** being produced shall meet either towards **b**, **d**, or towards **a**, **c**: let them be produced and meet towards **b**, **d** in the point **g**; therefore **g e f** is a triangle, and its exterior angle **a e f** is greater (i. 16) than the interior and opposite angle **e f g**; but it is also equal to it, which is impossible; therefore **a b** and **c d** being produced do not meet towards **b**, **d**. In like manner it may be demonstrated, that they do not meet towards **a**, **c**, but those straight lines which meet neither way, though produced ever so far, are parallel (35 *def.*) to one another. **a b** therefore is parallel to **c d**. Wherefore, if a straight line, &c. Q. E. D.



PROPOSITION XXVIII.—THEOREM.

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.



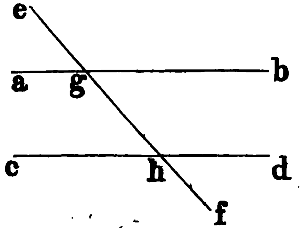
Let the straight line **e f**, which falls upon the two straight lines **a b**, **c d**, make the exterior angle **e g b** equal to the interior and opposite angle **g h d** upon the same side; or make the interior angles on the same side **b g h**, **g h d** together equal to two right angles; **a b** is parallel to **c d**.

Because the angle **e g b** is equal to the angle **g h d**, and the angle **e g b** equal (i. 15) to the angle **a g h**, the angle **a g h** is equal to the angle **g h d**; and they are the alternate angles; therefore **a b** is parallel (i. 27) to **c d**. Again, because the angles **b g h**, **g h d** are equal (*by hyp.*) to two right angles; and that **a g h**, **b g h**, are also equal (i. 13) to two right angles; the angles **a g h**, **b g h**, are equal to the angles **b g h**, **g h d**: take away the common angle **b g h**; therefore the remaining angle **a g h** is equal to the remaining angle **g h d**; and they are alternate angles; therefore **a b** is parallel to **c d**. Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXIX.—THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

LET the straight line **e f** fall upon the parallel straight lines **a b**, **c d**; the alternate angles **a g h**, **g h d**, are equal to one another; and the exterior angle **e g b** is equal to the interior and opposite, upon the same side **g h d**; and the two interior angles **b g h**, **g h d** upon the same side, are together equal to two right angles.

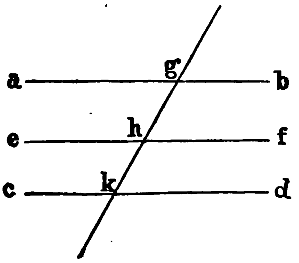


For, if **a g h** be not equal to **g h d**, one of them must be greater than the other; let **a g h** be the greater; and because the angle **a g h** is greater than the angle **g h d**, add to each of them the angle **b g h**; therefore the angles **a g h**, **b g h** are greater than the angles **b g h**, **g h d**; but the angles **a g h**, **b g h**, are equal (i. 13) to two right angles; therefore the angles **b g h**, **g h d**, are less than two right angles; but those straight lines which, with another straight line falling upon them, make the interior angles on the same side less than two right angles, do meet (12 *ax.*) together if continually produced; therefore the straight lines **a b**, **c d**, if produced far enough, shall meet; but they never meet, since they are parallel by the hypothesis; therefore the angle **a g h** is not unequal to the angle **g h d**, that is, it is equal to it; but the angle **a g h** is equal (i. 15) to the angle **e g b**; therefore likewise **e g b** is equal to **g h d**; add to each of these the angle **b g h**; therefore the angles **e g b**, **b g h** are equal to the angles **b g h**, **g h d**; but **e g b**, **b g h** are equal (i. 13) to two right angles; therefore also **b g h**, **g h d** are equal to two right angles. Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXX.—THEOREM.

Straight lines which are parallel to the same straight line are parallel to each other.

LET **a b**, **c d** be each of them parallel to **e f**; **a b** is also parallel to **c d**.

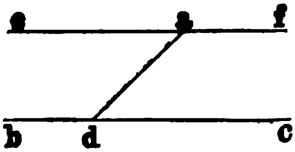


Let the straight line **g h k** cut **a b**, **e f**, **c d**; and because **g h k** cuts the parallel straight lines **a b**, **e f**, the angle **a g h** is equal (i. 29) to the angle **g h f**. Again, because the straight line **g k** cuts the parallel straight lines **e f**, **c d**, the angle **g h f** is equal (i. 29) to the angle **g k d**; and it was shewn that the angle **a g k** is equal to the angle **g h f**; therefore also **a g k** is equal to **g k d**; and they are alternate angles; therefore **a b** is parallel (i. 27) to **c d**; Wherefore, straight lines, &c. Q. E. D.

PROPOSITION XXXI.—PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

LET **a** be the given point, and **b c** the given straight line; it is required to draw a straight line through the point **a**, parallel to the straight line **b c**.



In **b c** take any point **d**, and join **a d**; and at the point **a**, in the straight line **a d**, make (i. 23) the angle **d a e** equal to the angle **a d c**; and produce the straight line **e a** to **f**.

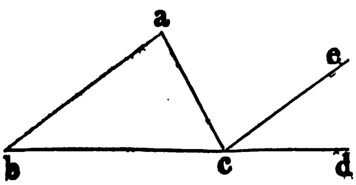
Because the straight line **a d**, which meets the two straight lines **b c**, **e f**, makes the alternate angles **e a d**, **a d c** equal to one another, **e f** is parallel (i. 27) to **b c**. Therefore the straight line **e a f** is drawn through the given point **a** parallel to the given straight line **b c**. Which was to be done.

PROPOSITION XXXII.—THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

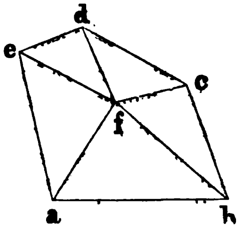
LET **a b c** be a triangle, and let one of its sides **b c** be produced to **d**; the exterior angle **a c d** is equal to the two interior and opposite angles **c a b**, **a b c**, and the three interior angles of the triangle, viz. **a b c**, **b c a**, **c a b**, are together equal to two right angles.

Through the point **c** draw **c e** parallel (i. 31) to the straight line **a b**; and because **a b** is parallel to **c e**, and **a c** meets them, the alternate angles **b a c**, **a c e**, are equal (i. 29). Again, because **a b** is parallel to **c e**, and **b d** falls upon them, the exterior angle **e c d** is equal to the interior and opposite angle **a b c**; but the angle **a c e** was shewn to be equal to the angle **b a c**; therefore the whole exterior angle **a c d** is equal to the two interior and opposite angles **c a b**, **a b c**; to these equals add the angle **a c b**, and the angles **a c d**, **a c b** are equal to the three angles **c b a**, **b a c**, **a c b**; but the angles **a c d**, **a c b** are equal (i. 13) to two right angles: therefore also the angles **c b a**, **b a c**, **a c b**, are equal to two right angles. Wherefore, if a side of a triangle, &c. Q. E. D.



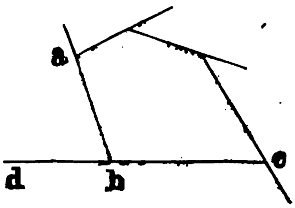
COR. 1. All the interior angles of any rectilineal figure, together with

four right angles, are equal to twice as many right angles as the figure has sides.



For any rectilineal figure **a b c d e**, can be divided into as many triangles as the figure has sides, by drawing straight lines from a point **f** within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles, that is, as there are sides of the figure: and the same angles are equal to the angles of the figure, together with the angles at the point **f**, which is the common vertex of the triangles: that is (i. 15, *cor.* 2), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.

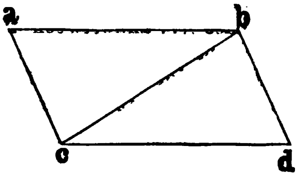


Because every interior angle **a b c**, with its adjacent exterior **a b d**, is equal (i. 13) to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

PROPOSITION XXXIII.—THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

Let **a b**, **c d** be equal and parallel straight lines, and joined towards the same parts by the straight lines **a c**, **b d**; **a c**, **b d** are also equal and parallel.



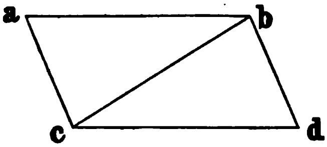
Join **b c**; and because **a b** is parallel to **c d**, and **b c** meets them, the alternate angles **a b c**, **b c d** are equal (i. 29); and because **a b** is equal to **c d**, and **b c** common to the two triangles **a b c**, **d c b**, the two sides **a b**, **b c**, are equal to the two **d c**, **c b**; and the angle **a b c** is equal to the angle **b c d**; therefore the base **a c** is equal (i. 4) to the base **b d**, and the triangle **a b c** to the triangle **b c d**, and the other angles to the other angles, (i. 4) each to each, to which the equal sides are opposite; therefore the angle **a c b** is equal to the angle **c b d**; and because the straight line **b c** meets the two straight lines **a c**, **b d**, and makes the alternate angles **a c b**, **c b d** equal to one another, **a c** is parallel (i. 27) to **b d**; and it was shewn to be equal to it. Therefore straight lines, &c. Q. E. D.

Proposition XXXIV.—Theorem.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects *them, that is, divides them into two equal parts.*

A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

Let **a c d b** be a parallelogram, of which **b c** is a diameter; the opposite sides and angles of the figure are equal to one another; and the diameter **b c** bisects it.



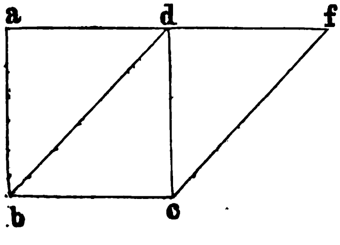
Because **a b** is parallel to **c d**, and **b c** meets them, the alternate angles **a b c**, **b c d** are equal (i. 29) to one another; and because **a c** is parallel to **b d**, and **b c** meets them, the alternate angles **a c b**, **c b d**, are equal (i. 29) to one another; wherefore the two triangles **a b c**, **c b d** have two angles **a b c**, **b c a** in one, equal to two angles **b c d**, **c b d** in the other, each to each, and one side **b c** common to the two triangles, which is adjacent to their equal angles; therefore their other sides shall be equal, each to each, and the third angle of the one to the third angle of the other (i. 26), viz. the side **a b** to the side **c d**, and **a c** to **b d**, and the angle **b a c** equal to the angle **b d c**: and because the angle **a b c** is equal to the angle **b c d**, and the angle **c b d** to the angle **a c b**, the whole angle **a b d** is equal to the whole angle **a c d**: and the angle **b a c** has been shewn to be equal to the angle **b d c**; therefore the opposite sides and angles of parallelograms are equal to one another; also, their diameter bisects them; for **a b** being equal to **c d**, and **b c** common, the two **a b**, **b c** are equal to the two **d c**, **c b**, each to each; and the angle **a b c** is equal to the angle **b c d**; therefore the triangle **a b c** is equal (i. 4) to the triangle **b c d**, and the diameter **b c** divides the parallelogram **a c d b** into two equal parts. Q. E. D.

Proposition XXXV.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

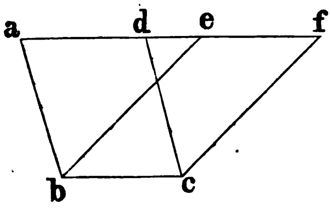
Let the parallelograms **a b c d**, **e b c f** (*see the second figure*) be upon the same base **b c**, and between the same parallels **a f**, **b c**; the parallelogram **a b c d** shall be equal to the parallelogram **e b c f**.

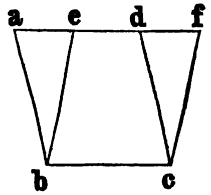
If the sides **a d**, **d f** of the parallelograms **a b c d**, **d b c f**, opposite to the base **b c**, be terminated in the same point **d**; it is plain that each of the parallelograms is double (i. 34) of the triangle **b d c**; and they are therefore equal to one another.



But, if the sides **a d**, **e f** (*see second and third figures*), opposite to the base **b c** of the parallelograms **a b c d**, **e b c f**, be not terminated in the same point **d**; then, because **a b c d** is a parallelogram, **a d** is equal (i. 34) to **b c**; for the same reason **e f** is equal to

bc; wherefore **ad** is equal (1 *ax.*) to **ef**; and **de** is common; therefore the whole, or the remainder, **ae** is equal (2 or 3 *ax.*) to the whole, or the remainder **df**; **ab** also is equal to **dc**; and the two **ea**, **ab** are therefore equal to the two **fd**, **dc**, each to each; and the exterior angle **fdc** is equal (i. 29) to the interior **eab**, therefore the base **eb** is equal to the base **fc**, and the triangle **eab** equal (i. 4) to the triangle **fdc**. Take the triangle **fdc** from the trapezium **abcf**, and from the same trapezium take the triangle **eab**: the remainders therefore are equal (3 *ax.*), that is, the parallelogram **abcd** is equal to the parallelogram **ebcf**. Therefore parallelograms upon the same base, &c. Q. E. D.



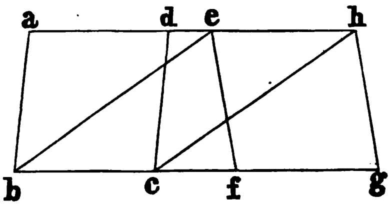


PROPOSITION XXXVI.—THEOREM.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

LET **abcd**, **efgh**, be parallelograms upon equal bases **bc**, **fg**, and between the same parallels **ah**, **bg**; the parallelogram **abcd** is equal to **efgh**.

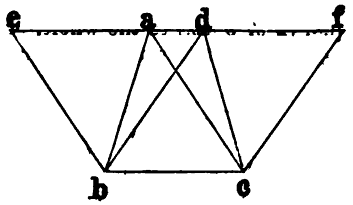
Join **be**, **ch**; and because **bc** is equal to **fg**, and **fg** to **eh** (i. 34), **bc** is equal to **eh**; and they are parallels, and joined towards the same parts by the straight lines **be**, **ch**. But straight lines which join equal and parallel straight lines towards the same parts, are themselves equal and parallel (i. 33); therefore **eb**, **ch**, are both equal and parallel, and **ebch** is a parallelogram; and it is equal (i. 35) to **abcd**, because it is upon the same base **bc**, and between the same parallels **bc**, **ad**. For the like reason, the parallelogram **efgh** is equal to the same **ebch**. Therefore also the parallelogram **abcd** is equal to **efgh**. Wherefore parallelograms, &c. Q. E. D.



PROPOSITION XXXVII.—THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

LET the triangles **abc**, **dbc**, be upon the same base **bc** and between



the same parallels **ad**, **bc**. The triangle **abc** is equal to the triangle **dbc**.

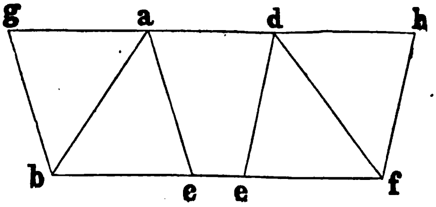
Produce **ad** both ways to the points **e**, **f**, and through **b** draw (i. 31) **be** parallel to **ca**; and through **c** draw **cf** parallel to **bd**; therefore each of the figures **ebca**, **dbcf** is a parallelogram; and **ebca** is equal (i. 35) to **dbcf**, because they are upon the same base **bc**, and between the same parallels **bc**, **ef**; and the triangle **abc** is the half of the parallelogram **ebca**, because the diameter **ab** bisects it (i. 34); and the triangle **dbc** is the half of the parallelogram **dbcf**, because the diameter **dc** bisects it: but the halves of equal things are equal (7 *ax.*); therefore the triangle **abc** is equal to the triangle **dbc**. Wherefore triangles, &c. Q. E. D.

PROPOSITION XXXVIII.—THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.

LET the triangles **abc**, **def** be upon equal bases **bc**, **ef**, and between the same parallels **bf**, **ad**. The triangle **abc** is equal to the triangle **def**.

Produce **ad** both ways to the points **g**, **h**, and through **b** draw **bg** parallel (i. 31) to **ca**, and through **f** draw **fh** parallel to **ed**. Then each of the figures **gbea**, **defh**, is a parallelogram; and they are equal (i. 36) to one another, because they are upon equal bases **bc**, **ef**, and between the same parallels **bf**, **gh**; and the triangle **abc** is the half (i. 34) of the parallelogram **gbea**, because the diameter **ab** bisects it; and the triangle **def** is the half (i. 34) of the parallelogram **defh**, because the diameter **df** bisects it: but the halves of equal things are equal (7 *ax.*); therefore the triangle **abc** is equal to the triangle **def**. Wherefore triangles, &c. Q. E. D.

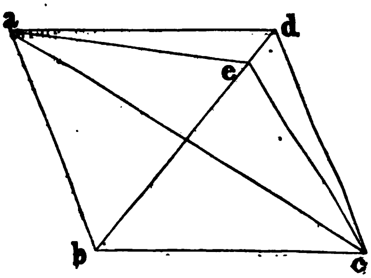


PROPOSITION XXXIX.—THEOREM.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

LET the equal triangles **abc**, **dbc** be upon the same base **bc**, and upon the same side of it; they are between the same parallels.

Join **ad**; **ad** is parallel to **bc**; for, if it is not, through the point **a** draw

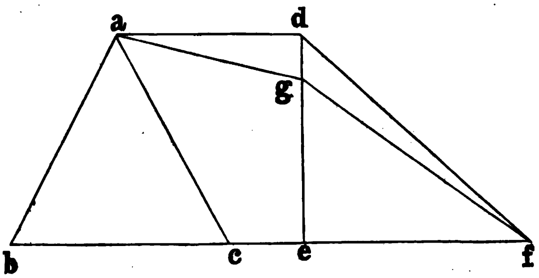


(i. 31) **a e** parallel to **b c**, and join **e c**. The triangle **a b c** is equal (i. 37) to the triangle **e b c**, because it is upon the same base **b c**, and between the same parallels **b c**, **a e**. But the triangle **a b c** is equal to the triangle **b d c**, therefore also the triangle **b d c** is equal to the triangle **e b c**, the greater to the less, which is impossible. Therefore **a e** is not parallel to **b c**. In the same manner, it can be demonstrated, that no other line but **a d** is parallel to **b c**; **a d** is therefore parallel to it. Wherefore equal triangles, &c. Q. E. D.

PROPOSITION XL.—THEOREM.

Equal triangles upon equal bases, in the same straight line and towards the same parts, are between the same parallels.

LET the equal triangles **a b c**, **d e f** be upon equal bases **b c**, **e f**, in the same straight line **b f**, and towards the same parts; they are between the same parallels.

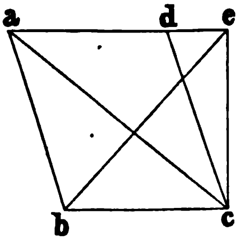


Join **a d**; **a d** is parallel to **b c**. For, if it is not, through **a** draw (i. 31) **a g** parallel to **b f**, and join **g f**: the triangle **a b c** is equal (i. 38) to the triangle **g e f**, because they are upon equal bases **b c**, **e f**, and between the same parallels **b f**, **a g**: but the triangle **a b c** is equal to the triangle **d e f**; therefore also the triangle **d e f** is equal to the triangle **g e f**, the greater to the less, which is impossible: Therefore **a g** is not parallel to **b f**. And in the same manner it can be demonstrated that there is no other parallel to it but **a d**: **a d** is therefore parallel to **b f**. Wherefore equal triangles, &c. Q. E. D.

PROPOSITION XLI.—THEOREM.

If a parallelogram and triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.

LET the parallelogram **a b c d** and the triangle **e b c** be upon the same



base **b c**, and between the same parallels **b c**, **a e**; the parallelogram **a b c d** is double of the triangle **e b c**.

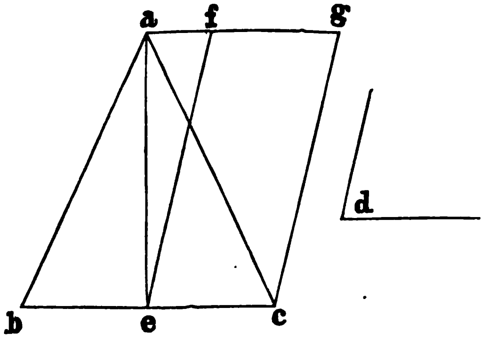
Join **a c**; then the triangle **a b c** is equal (i. 37) to the triangle **e b c**, because they are upon the same base **b c**, and between the same parallels **b c**, **a e**. But the parallelogram **a b c d** is double (i. 34) of the triangle **a b c**, because the diameter **a c** divides it into two equal parts; wherefore **a b c d** is also double of the triangle **e b c**. Therefore, if a parallelogram, &c. Q. E. D.

Proposition XLII.—Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

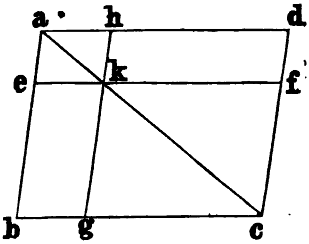
Let **a b c** be the given triangle, and **d** the given rectilineal angle. It is required to describe a parallelogram that shall be equal to the given triangle **a b c**, and have one of its angles equal to **d**.

Bisect (i. 10) **b c** in **e**, join **a e**, and at the point **e** in the straight line **e c** make (i. 23) the angle **c e f** equal to **d**; and through **a** draw (i. 31) **a g** parallel to **e c**, and through **c** draw **c g** parallel to **e f**: therefore **f e c g** is a parallelogram. And because **b e** is equal to **e c**, the triangle **a b e** is likewise equal (i. 38) to the triangle **a e c**, since they are upon equal bases **b e**, **e c**, and between the same parallels, **b c**, **a g**; therefore the triangle **a b c** is double of the triangle **a e c**. And the parallelogram **f e c g** is likewise double (i. 41) of the triangle **a e c**, because it is upon the same base, and between the same parallels: therefore the parallelogram **f e c g** is equal to the triangle **a b c**, and it has one of its angles **c e f** equal to the given angle **d**; wherefore there has been described a parallelogram **f e c g** equal to a given triangle **a b c**, having one of its angles **c e f** equal to the given angle **d**. Which was to be done.



PROPOSITION XLIII.—THEOREM.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.



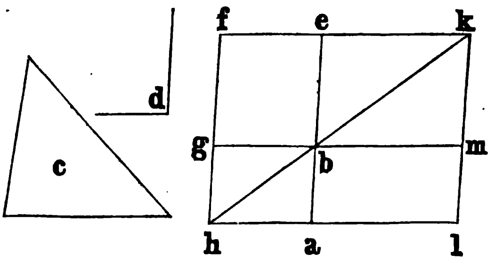
LET **abcd** be a parallelogram, of which the diameter is **ac**, and **eh**, **fg**, the parallelograms about **ac**, *that is, through which* **ac** *passes*, and **bk**, **kd**, the other parallelograms which make up the whole figure **abcd**, which are therefore called the complements. The complement **bk** is equal to the complement **kd**.

Because **abcd** is a parallelogram, and **ac** its diameter, the triangle **abc** is equal (i. 34) to the triangle **adc**: and, because **ekha** is a parallelogram, the diameter of which is **ak**, the triangle **aek** is equal to the triangle **ahk**: for the same reason, the triangle **kgc** is equal to the triangle **kfc**. Then, because the triangle **aek** is equal to the triangle **ahk**, and the triangle **kgc** to **kfc**; the triangle **aek**, together with the triangle **kgc** is equal to the triangle **ahk** together with the triangle **kfc**. But the whole triangle **abc** is equal to the whole **adc**; therefore the remaining complement **bk** is equal to the remaining complement **kd**. Wherefore the complements, &c. Q. E. D.

PROPOSITION XLIV.—PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

LET **ab** be the given straight line, and **c** the given triangle, and **d** the given rectilineal angle. It is required to apply to the straight line **ab** a parallelogram equal to the triangle **c**, and having an angle equal to **d**.



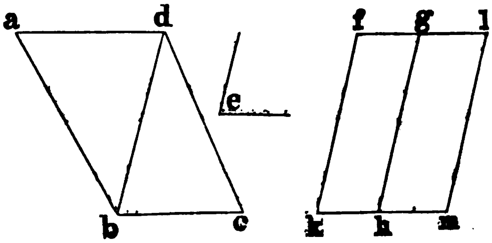
Make (i. 42) the parallelogram **befg** equal to the triangle **c**, and having the angle **ebg** equal to the angle **d**, so that **be** be in the same straight line with **ab**, and produce **fg** to **h**; and through **a** draw (i. 31) **ah** parallel to **bg** or **ef**, and join **hb**. Then, because the straight line **hf** falls upon the parallels **ah**, **ef**, the angles **ahf**, **hfe**, are together equal (i. 29) to two right angles; wherefore the angles **bhf**, **hfe**, are less than two right angles. But straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet (12 *ax.*) if produced far enough: therefore **hb**, **fe** shall meet if produced; let them meet in **k**, and through **k** draw **k l**, parallel to **e a** or **f h**, and produce **h a**, **g b** to the points **l**, **m**: then **h l k f** is a parallelogram, of which the diameter is **h k**, and **a g**, **m e** are the parallelograms about **h k**; and **l b**, **b f** are the complements: therefore **l b** is equal (i. 43) to **b f**; but **b f** is equal to the triangle **c**; wherefore **l b** is equal to the triangle **c**; and because the angle **g b e** is equal (i. 15) to the angle **a b m**, and likewise to the angle **d**; the angle **a b m** is equal to the angle **d**. Therefore the parallelogram **l b** is applied to the straight line **a b**, is equal to the triangle **c**, and has the angle **a b m** equal to the angle **d**. Which was to be done.

PROPOSITION XLV.—PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

LET **a b c d** be the given rectilineal figure, and **e** the given rectilineal angle. It is required to describe a parallelogram equal to **a b c d**, and having an angle equal to **e**.

Join **d b**, and describe (i. 42) the parallelogram **f h** equal to the triangle **a d b**, and having the angle **h k f** equal to the angle **e**; and to the straight line **g h** apply (i. 44) the parallelogram **g m** equal to the triangle **d b c**, having the angle **g h m** equal to the angle **e**; and because the angle **e** is equal to each of the angles **f k h**, **g h m**, the angle **f k h** is equal to **g h m**: add to each of these the angle **k h g**; therefore the angles **f k h**, **k h g**, are equal to the angles **k h g**, **g h m**; but **f k h**, **k h g** are equal (i. 29) to two right angles; therefore also **k h g**, **g h m**, are equal to two right angles; and because at the point **h** in the straight line **g h**, the two straight lines **k h**, **h m** upon the opposite sides of it make the adjacent angles equal to two right angles, **k h** is in the same straight line (i. 14) with **h m**; and because the straight line **h g** meets the parallels **k m**, **f g**, the alternate angles **m h g**, **h g f** are equal (i. 29): add to each of these the angle **h g l**: therefore the angles **m h g**, **h g l**, are equal to the angles **h g f**, **h g l**. But the angles **m h g**, **h g l**, are equal (i. 29) to two right angles; wherefore also the angles **h g f**, **h g l** are equal to two right angles, and **f g** is therefore in the same straight line with **g l**; and because **k f** is parallel to **h g**, and **h g** to **m l**; **k f** is parallel (i. 30) to **m l**; and **k m**, **f l** are parallels; wherefore **k f l m** is a parallelogram; and because the triangle **a b d** is equal to the parallelogram **h f**, and the triangle **d b c** to the parallelogram **g m**; the whole rectilineal figure **a b c d** is equal to the whole parallelogram **k f l m**; therefore the parallelogram **k f l m** has been described equal to the given rectilineal figure **a b c d**, having the angle **f k m** equal to the given angle **e**. Which was to be done.



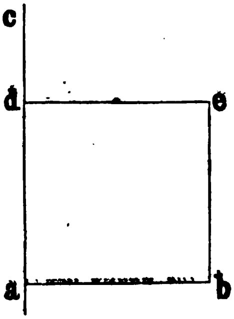
COR. From this it is manifest how to a given straight line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure, viz. by applying (i. 44) to the given straight line a parallelogram equal to the first triangle a b d, and having an angle equal to the given angle.

PROPOSITION XLVI.—PROBLEM.

To describe a square upon a given straight line.

LET a b be the given straight line; it is required to describe a square upon a b.

From the point a draw (i. 11) a c at right angles to a b; and make (i. 3) a d equal to a b, and through the point d draw d e parallel (i. 31) to a b, and through b draw b e parallel to a d therefore a d e b is a parallelogram: whence a b is equal (i. 34) to d e, and a d to b e: but b a is equal to a d; therefore the four straight lines b a, a d, d e, e b, are equal to one another, and the parallelogram a d e b is equilateral, likewise all its angles are right angles; because the straight line a d meeting the parallels a b, d e, the angles b a d, a d e are equal (i. 29) to two right angles: but b a d is a right angle; therefore also a d e is a right angle; but the opposite angles of parallelograms are equal (i. 34); therefore each of the opposite angles a b e, b e d is a right angle; wherefore the figure a d e b is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line a b. Which was to be done.



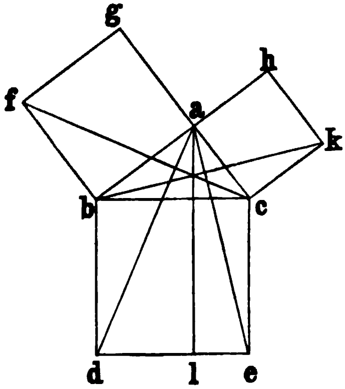
COR. Hence every parallelogram that has one right angle has all its angles right angles.

PROPOSITION XLVII.—THEOREM.

In any right-angled triangle, the square which is described upon the side subtending the right angle is equal to the squares described upon the sides which contain the right angle.

LET a b c be a right-angled triangle having the right angle b a c; the square described upon the side b c is equal to the squares described upon b a, a c.

On b c describe (i. 46) the square b d e c, and on b a, a c the squares g b, h c; and through a draw (i. 31) a l parallel to b d, or c e, and join a d, f c. Then, because each of the angles b a c, b a g is a right angle (30 *def.*), the two straight lines a c, a g, upon the opposite sides of a b, make with it at the point a the adjacent angles equal to two right angles; therefore c a is in the same straight line (i. 14) with a g; for the same reason, a b and a h are in the same straight line; and because the angle d b c is equal to the angle f b a, each of them being a right angle, add to each the angle a b c, and the whole angle d b a is equal (2 *ax.*) to the whole **fbc**; and because the two sides **ab**, **bd**, are equal to the two **fb**, **bc**, each to each, and the angle **dba** equal to the angle **fbc**; therefore the base **ad** is equal (i. 4) to the base **fc**, and the triangle **abd** to the triangle **fbc**: now the parallelogram **bl** is double (i. 41) of the triangle **abd**, because they are upon the same base **bd**, and between the same parallels **bd**, **al**; and the square **gb** is double of the triangle **fbc**, because these also are upon the same base **fb**, and between the same parallels **fb**, **gc**. But the doubles of equals are equal (6 *ax.*) to one another: therefore the parallelogram **bl** is equal to the square **gb**: and, in the same manner, by joining **ae**, **bk**, it is demonstrated, that the parallelogram **cl** is equal to the square **hc**; therefore the whole square **bdec** is equal to the two squares **gb**, **hc**; and the square **bdec** is described upon the straight line **bc**, and the squares **gb**, **hc** upon **ba**, **ac**: wherefore the square upon the side **bc** is equal to the squares upon the sides **ba**, **ac**. Therefore, in any right-angled triangle, &c. Q. E. D.

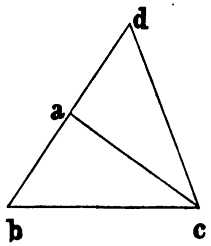


Proposition XLVIII.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

If the square described upon **bc**, one of the sides of the triangle **abc**, be equal to the squares upon the other sides **ba**, **ac**, the angle **bac** is a right angle.

From the point **a** draw (i. 11) **ad** at right angles to **ac**, and make **ad** equal to **ba**, and join **dc**: Then, because **da** is equal to **ab**, the square of **da** is equal to the square of **ab**. To each of these add the square of **ac**; therefore the squares of **da**, **ac** are equal to the squares of **ba**, **ac**. But the square of **dc** is equal (i. 47) to the squares of **da**, **ac**, because **dac** is a right angle; and the square of **bc**, by hypothesis, is equal to the squares of **ba**, **ac**; therefore the square of **dc** is equal to the square of **bc**; and therefore also the side **dc** is equal to the side **bc**. And because the side **da** is equal to **ab**, and **ac** common to the two triangles **dac**, **bac**, the two **da**, **ac** are equal to the two **ba**, **ac**; and the base **dc** is equal to the base **bc**; therefore the angle **dac** is equal (i. 8) to the angle **bac**; but **dac** is a right angle; therefore also **bac** is a right angle. Therefore, if the square, &c. Q. E. D.



EXERCISES ON BOOK I.

SECT. I.—PROBLEMS.

1. Given a straight line and two points, to find a point in the line equidistant from each of the given points.

2. Given a straight line, a point, and an angle, to draw from the point to the line another straight line that shall make with it an angle equal to the given angle.

3. To divide a given finite straight line into any given number of equal parts.

4. To trisect a right angle, that is, to divide it into three equal parts.

5. Given an angle divided into any number of equal angles, to divide one half of the angle into the same number of equal angles.

6. Given the base, one of the angles at the base, and the sum of the other two sides, to construct the triangle.

7. To find a point in a triangle equidistant from the three angular points of the triangle.

8. Given the three angles of a triangle and the sum of its three sides to construct the triangle.

9. To find a square which shall be equal to any number of given squares.

10. To bisect a parallelogram by a straight line drawn *through a given* point in one of its sides.

11. To inscribe a square in a parallelogram.

12. Given a right-angled isosceles triangle to inscribe in it a square.

13. Given a finite straight line to describe a square of which that line shall be the diameter.

14. To construct a triangle that shall be equal to any given rectilineal figure.

SECT. II.—THEOREMS.

15. There cannot be drawn more than two equal straight lines to another straight line from a given point without it.

16. The straight line drawn from a given point perpendicular to a given line is the shortest that can be drawn from the point to the line.

17. If a triangle have its exterior angle and one of its opposite interior angles double of those in another triangle, its remaining opposite interior angle is also double of the corresponding angle in the other.

18. If two sides of a triangle be bisected, the straight line joining the two points of bisection will be parallel to the other side.

19. The difference between the two sides of a triangle is less than the third side.

20. The sum of two sides of a triangle is greater than twice the line joining the vertex and the middle of the base.

21. Each angle at the base of an isosceles triangle is equal to, or is less, or is greater than the half of the vertical angle, according as the triangle is right, obtuse, or acute-angled.

22. If the vertical angle of an isosceles triangle be bisected, the bisecting line will bisect the base, and be perpendicular to it.

23. In an isosceles triangle, if either of the equal sides be produced beyond the vertex, the line that bisects the exterior angle will be parallel to the base.

24. The diameters of a rectangle are equal to one another.

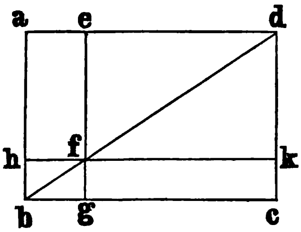
25. If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

26. In a parallelogram the diameters bisect each other.

BOOK II.

DEFINITIONS.

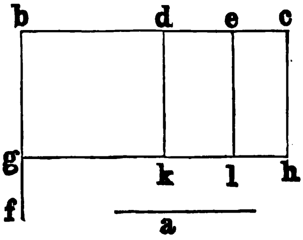
I. EVERY right-angled parallelogram is said to be contained by any two of the straight lines which contain one of the right angles.



II. In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a *gnomon*. Thus the parallelogram **h g**, together with the complements **a f**, **f c**, is the gnomon, which is more briefly expressed by the letters **a g k**, or **e h c**, which are at the opposite angles of the parallelograms which make the gnomon.

PROPOSITION I.—THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.



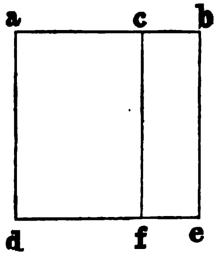
LET **a** and **b c** be two straight lines; and let **b c** be divided into any parts in the points **d**, **e**; the rectangle contained by the straight lines **a**, **b c** is equal to the rectangle contained by **a**, **b d**, together with that contained by **a**, **d e**, and that contained by **a**, **e c**.

From the point **b** draw (i. 11) **b f** at right angles to **b c**, and make **b g** equal (i. 3) to **a**; and through **g** draw (i. 31) **g h** parallel to **b c**; and through **d**, **e**, **c**, draw (i. 31) **d k**, **e l**, **c h** parallel to **b g**; then the rectangle **b h** is equal to the rectangles **b k**, **d l**, **e h**; and **b h** is contained by **a**, **b c**, for it is contained by **g b**, **b c**, and **g b** is equal to **a**; and **b k** is contained by **a**, **b d**, for it is contained by **g b**, **b d**, of which **g b** is equal to **a**; and **d l** is contained by **a**, **d e**, because **d k**, that is (i. 34) **b g** is equal to **a**; and in like manner the rectangle **e h** is contained by **a**, **e c**. Therefore the rectangle contained by a, bc, is equal to the several rectangles contained by a, bd, and by a, de, and also by a, ec. Wherefore, if there be two straight lines, &c. Q. E. D.

Proposition II.—Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square of the whole line.

Let the straight line ab be divided into any two parts in the point c; the rectangle contained by ab, bc, together with the rectangle* ab, ac, shall be equal to the square of ab.

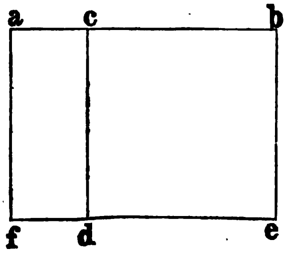


Upon ab describe (i. 46) the square adeb, and through c draw (i. 31) cf, parallel to ad or be. Then ae is equal to the rectangles af, ce; and ae is the square of ab; and af is the rectangle contained by ba, ac; for it is contained by da, ac, of which ad is equal to ab; and ce is contained by ab, bc, for be is equal to ab; therefore the rectangle contained by ab, ac, together with the rectangle ab, bc, is equal to the square of ab. If therefore a straight line, &c. Q. E. D.

Proposition III.—Theorem.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line ab be divided into any two parts in the point c; the rectangle ab, bc is equal to the rectangle ac, cb, together with the square of bc.



Upon bc describe (i. 46) the square cdeb, and produce ed to f, and through a draw (i. 31) af parallel to cd or be; then the rectangle ae is equal to the rectangles ad, ce; and ae is the rectangle contained by ab, bc, for it is contained by ab, be, of which be is equal to bc; and ad is contained by ac, cb, for cd is equal to cb; and db is the square of bc; therefore the rectangle ab, bc, is equal to the rectangle ac, cb, together with the square of bc. If therefore a straight line, &c. Q. E. D.

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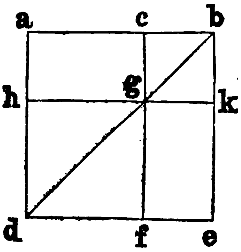
* To avoid repeating the word *contained* too frequently, the rectangle contained by two straight lines ab, ac is sometimes simply called the rectangle ab, ac.

Proposition IV.—Theorem.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let the straight line **a b** be divided into any two parts in **c**; the square of **a b** is equal to the squares of **a c, c b**, and to twice the rectangle contained by **a c, c b**.

Upon **a b** describe (i. 46) the square **a d e b**, and join **b d**, and through **c** draw (i. 31) **c g f** parallel to **a d** or **b e**, and through **g** draw **h k** parallel to **a b** or **d e**. And because **c f** is parallel to **a d**, and **b d** falls upon them, the exterior angle **b g c** is equal (i. 29) to the interior and opposite angle **a d b**; but **a d b** is equal (i. 5) to the angle **a b d**, because **b a** is equal to **a d**, being sides of a square; wherefore the angle **c g b** is equal to the angle **g b c**; and therefore the side **b c** is equal (i. 6) to the side **c g**. But **c b** is equal (i. 34) also to **g k**, and **c g** to **b k**; wherefore the figure **c g k b** is equilateral. It is likewise rectangular; for **c g** is parallel to **b k**, and **c b** meets them; the angles **k b c**, **g c b** are therefore equal to two right angles; and **k b c** is a right angle; wherefore **g c b** is a right angle: and therefore also the angles (i. 34) **c g k, g k b**, opposite to these, are right angles, and **c g k b** is rectangular; but it is also equilateral, as was demonstrated; wherefore it is a square, and it is upon the side **c b**. For the same reason **h f** also is a square, and it is upon the side **h g**, which is equal to **a c**. Therefore **h f, c k** are the squares of **a c, c b**; and because the complement **a g** is equal (i. 43) to the complement **g e**, and that **a g** is the rectangle contained by **a c, c b**, for **g c** is equal to **c b**; therefore **g e** is also equal to the rectangle **a c, c b**; wherefore **a g, g e** are equal to twice the rectangle **a c, c b**. And **h f, c k** are the squares of **a c, c b**; wherefore the four figures **h f, c k, a g, g e** are equal to the squares of **a c, c b**, and to twice the rectangle **a c, cb**. But **h f, c k, a g, g e** make up the whole figure **a d e b**, which is the square of **a b**. Therefore the square of **a b** is equal to the squares of **a c, c b**, and twice the rectangle **a c, c b**. Wherefore, if a straight line, &c. Q. E. D.



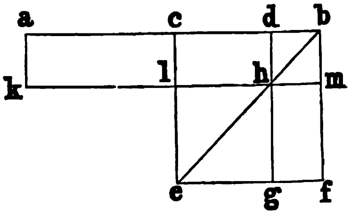
Cor. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

Proposition V.—Theorem.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let the straight line **a b** be divided into two equal parts in the point **c**, and into two unequal parts at the point **d**; the rectangle **a d**, **d b**, together with the square of **c d**, is equal to the square of **c b**.

Upon **c b** describe (i. 46) the square **c e f b**, join **b e**, and through **d** draw (i. 31) **d h g** parallel to **c e** or **b f**; and through **h** draw **k l m** parallel to **c b** or **e f**; and also through **a** draw **a k** parallel to **c l** or **b m**. And because the complement **c h** is equal (i. 43) to the complement **h f**, to each of these add **d m**; therefore the whole **c m** is equal to the whole **d f**; but **c m** is equal (i. 36) to **a l**, because **a c** is equal to **c b**; therefore also **a l** is equal to **d f**. To each of these add **c h**, and the whole **a h** is equal to **d f** and **c h**: but **a h** is the rectangle contained by **a d**, **d b**, for **d h** is equal (ii. 4. *cor.*) to **d b**; and **d f** together with **c h** is the gnomon **c m g**; therefore the gnomon **c m g** is equal to the rectangle **a d**, **d b**: to each of these add **l g**, which is equal (ii. 4. *cor.*) to the square of **c d**; therefore the gnomon **c m g**, together with **l g**, is equal to the rectangle **a d**, **d b**, together with the square of **c d**; but the gnomon **c m g** and **l g** make up the whole figure **c e f b**, which is the square of **c b**: therefore the rectangle **a d**, **d b**, together with the square of **c d**, is equal to the square of **c b**. Wherefore, if a straight line, &c. Q. E. D.



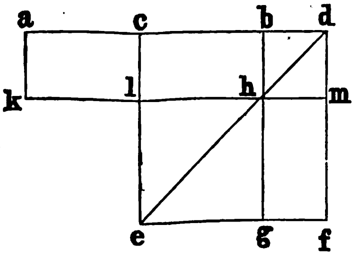
Cor. From this proposition it is manifest, that the difference of the squares of two unequal lines **a c**, **c d**, is equal to the rectangle contained by their sum and difference.

Proposition VI.—Theorem.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half and the part produced.

Let the straight line **a b** be bisected in **c**, and produced to the point **d**; the rectangle **a d**, **d b**, together with the square of **c b**, is equal to the square of **c d**.

Upon **c d** describe (i. 46) the square **c e f d**, join **d e**, and through **b** draw (i. 31) **b h g** parallel to **c e** or **d f**, and through **h** draw **k l m** parallel to **a d** or **e f**, and also through **a** draw **a k** parallel to **c l** or **d m**; and because **a c** is equal to **c b**, the rectangle **a l** is equal (i. 36) to **c h**; but **c h** is equal (i. 43) to **h f**; therefore also **a l** is equal to **h f**: to each of these add **c m**; therefore the whole **a m** is equal to the gnomon **c m g**: and **a m** is the rectangle contained by **a d**, **d b**, for **d m** is equal (ii. 4. *cor.*) to **d b**: therefore the gnomon **c m g** is equal to the rectangle **a d**, **d b**: add to each of these **l g**, which is equal to the square of **c b**; therefore the rect-



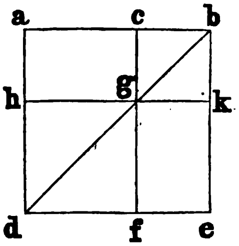
angle **a d**, **d b**, together with the square of **c b**, is equal to the gnomon **c m g**, and the figure **l g**; but the gnomon **c m g** and **l g** make up the whole figure **c e f d**, which is the square of **c d**; therefore the rectangle **a d**, **d b**, together with the square of **c b**, is equal to the square of **c d**. Wherefore, if a straight line, &c. Q. E. D.

Proposition VII.—Theorem.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the straight line **a b** be divided into any two parts in the point **c**; the squares of **a b**, **b c** are equal to twice the rectangle **a b**, **b c**, together with the square of **a c**.

Upon **a b** describe (i. 46) the square **a d e b**, and construct the figure as in the preceding propositions; and because **a g** is equal (i. 43) to **g e**, add to each of them **c k**; the whole **a k** is therefore equal to the whole **c e**; therefore **a k**, **c e**, are double of **a k**: but **a k**, **c e** are the gnomon **a k f**, together with the square **c k**; therefore the gnomon **a k f**, together with the square **c k**, is double of **a k**: but twice the rectangle **a b**, **b c** is double of **a k**, for **b k** is equal (ii. 4. *cor.*) to **b c**: therefore the gnomon **a k f**, together with the square **c k**, is equal to twice the rectangle **a b**, **b c**: to each of these equals add **h f**, which is equal to the square of **a c**; therefore the gnomon **a k f**, together with the squares **c k**, **h f**, is equal to twice the rectangle **a b**, **b c**, and the square of **a c**: but the gnomon **a k f**, together with the squares **c k**, **h f**, make up the whole figure **a d e b** and **c k**, which are the squares of **a b** and **b c**: therefore the squares of **a b** and **b c** are equal to twice the rectangle **a b**, **b c**, together with the square of **a c**. Wherefore, if a straight line, &c. Q. E. D.

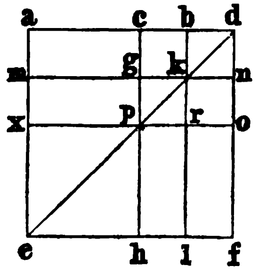


Proposition VIII.—Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the straight line which is made up of the whole and that part.

Let the straight line **a b** be divided into any two parts in the point **c**; four times the rectangle **a b**, **b c**, together with the square of **a c**, is equal to the square of the straight line made up of **a b** and **b c** together.

Produce **a b** to **d**, so that **b d** be equal to **c b**, and upon **a d** describe the square **a e f d**; and construct two figures such as in the preceding. Because **c b** is equal to **b d**, and that **c b** is equal (i. 34) to **g k**, and **b d** to **k n**; therefore **g k** is equal to **k n**: for the same reason, **p r** is equal to r o; and because c b is equal to b d, and g k to k n, the rectangle c k is equal (i. 36) to b n, and g r to r n; but c k is equal (i. 43) to r n, because they are the complements of the parallelogram c o; therefore also b n is equal to g r; and the four rectangles b n, c k, g r, r n are therefore equal to one another, and so are quadruple of one of them c k. Again, because c b is equal to b d, and that b d is equal (ii. 4. *cor.*) to b k, that is, to c g, and c b equal to g k, that is, to g p (ii. 4. *cor.*); therefore c g is equal to g p: and because c g is equal to g p, and p r to r o, the rectangle a g is equal to m p, and p l to r f: but m p is equal (i. 43) to p l, because they are the complements of the parallelogram m l; wherefore a g is equal also to r f: therefore the four rectangles a g, m p, p l, r f, are equal to one another, and so are quadruple of one of them a g. And it was demonstrated that the four c k, b n, g r, and r n are quadruple of c k. Therefore the eight rectangles which contain the gnomon a o h are quadruple of a k; and because a k is the rectangle contained by a b, b c, for b k is equal to b c, four times the rectangle a b, b c is quadruple of a k: but the gnomon a o h was demonstrated to be quadruple of a k; therefore four times the rectangle a b, b c, is equal to the gnomon a o h. To each of these add x h, which is equal (ii. 4. *cor.*) to the square of a c: therefore four times the rectangle a b, b c, together with the square of a c, is equal to the gnomon a o h and the square x h: but the gnomon a o h and x h make up the figure a e f d, which is the square of a d: therefore four times the rectangle a b, b c, together with the square of a c, is equal to the square of a d, that is, of a b and b c added together in one straight line. Wherefore, if a straight line, &c. Q. E. D.

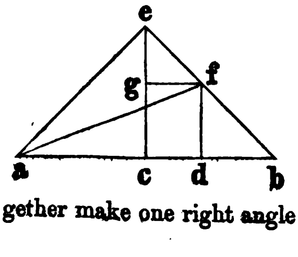


PROPOSITION IX.—THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of section.

LET the straight line a b be divided at the point c into two equal, and at d into two unequal parts: the squares of a d, d b are together double of the squares of a c, c d.

From the point c draw (i. 11) c e at right angles to a b, and make it equal to a c or c b, and join e a, e b; through d draw (i. 31) d f parallel to c e, and through f draw f g parallel to a b; and join a f: then, because a c is equal to c e, the angle e a c is equal (i. 5) to the angle a e c; and because the angle a c e is a right angle, the two others a e c, e a c together make one right angle (i. 32); and they are equal to one another;



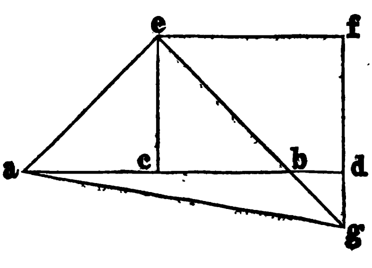
each of them therefore is half of a right angle. For the same reason, each of the angles **ceb**, **ebc** is half a right angle; and therefore the whole **aeb** is a right angle: and because the angle **gef** is half a right angle, and **egf** a right angle, for it is equal (i. 29) to the interior and opposite angle **ecb**, the remaining angle **efg** is half a right angle; therefore the angle **gef** is equal to the angle **efg**, and the side **eg** equal (i. 6) to the side **gf**: again, because the angle at **b** is half a right angle, and **fdb** a right angle, for it is equal (i. 29) to the interior and opposite angle **ecb**, the remaining angle **bfd** is half a right angle; therefore the angle at **b** is equal to the angle **bfd**, and the side **df** to (i. 6) the side **db**: and because **ac** is equal to **ce**, the square of **ac** is equal to the square of **ce**; therefore the squares of **ac**, **ce**, are double of the square of **ac**: but the square of **ea** is equal (i. 47) to the squares of **ac**, **ce**, because **ace** is a right angle; therefore the square of **ea** is double of the square of **ac**: again, because **eg** is equal to **gf**, the square of **eg** is equal to the square of **gf**; therefore the squares of **eg**, **gf** are double of the square of **gf**; but the square of **ef** is equal to the squares of **eg**, **gf**; therefore the square of **ef** is double of the square **gf**; and **gf** is equal (i. 34) to **cd**; therefore the square of **ef** is double of the square of **cd**: but the square of **ae** is likewise double of the square of **ac**; therefore the squares of **ae**, **ef** are double of the squares of **ac**, **cd**: and the square of **af** is equal (i. 47) to the squares of **ae**, **ef**, because **aef** is a right angle; therefore the square of **af** is double of the squares of **ac**, **cd**: but the squares of **ad**, **df**, are equal to the square of **af**, because the angle **adf** is a right angle; therefore the squares of **ad**, **df** are double of the squares of **ac**, **cd**: and **df** is equal to **db**; therefore the squares of **ad**, **db** are double of the squares of **ac**, **cd**. If therefore a straight line, &c. Q. E. D.

PROPOSITION X.—THEOREM.

If a straight line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

LET the straight line **ab** be bisected in **c** and produced to the point **d**; the squares of **ad**, **db** are double of the squares of **ac**, **cd**.

From the point **c** draw (i. 11) **ce** at right angles to **ab**: and make it equal to **ac** or **cb**, and join **ae**, **eb**; through **e** draw (i. 31) **ef** parallel to **ab**, and through **d** draw **df** parallel to **ce**: and because the straight line **ef** meets the parallels **ec**, **fd**, the angles **cef**, **efd** are equal (i. 29) to two right angles; and therefore the angles **bef**, **efd** are less than two right angles; but straight lines which with another straight line make the interior angles upon the same side less than two right angles, do meet (12 *ax.*) if produced far enough: therefore **eb**, **fd** shall meet if produced towards **b**, **d**: let them meet in **g**, and join **ag**: then,



because ac is equal to ce, the angle cea is equal (i. 5) to the angle eac; and the angle ace is a right angle; therefore each of the angles cea, eac is half a right angle (i. 32). For the same reason, each of the angles ceb, ebc is half a right angle; therefore aeb is a right angle: and because ebc is half a right angle, dbg is also (i. 15) half a right angle, for they are vertically opposite; but bdg is a right angle, because it is equal (i. 29) to the alternate angle dce; therefore the remaining angle dgb is half a right angle, and is therefore equal to the angle dbg; wherefore also the side bd is equal (i. 6) to the side dg. Again, because egf is half a right angle, and that the angle at f is a right angle, because it is equal (i. 34) to the opposite angle ecd, the remaining angle feg is half a right angle, and equal to the angle egf; wherefore also the side gf is equal (i. 6) to the side fe. And because ec is equal to ca, the square of ec is equal to the square of ca; therefore the squares of ec, ca are double of the square of ca. But the square of ea is equal (i. 47) to the squares of ec, ca; therefore the square of ea is double of the square of ac. Again, because gf is equal to fe, the square of gf is equal to the square of fe; and therefore the squares of gf, fe are double of the square of ef: but the square of eg is equal (i. 47) to the squares of gf, fe; therefore the square of eg is double of the square of ef: and ef is equal to cd; wherefore the square of eg is double of the square of cd. But it was demonstrated, that the square of ea is double of the square of ac: therefore the squares of ae, eg, are double of the squares of ac, cd: and the square of ag is equal (i. 47) to the squares of ae, eg; therefore the square of ag is double of the squares of ac, cd: but the squares of ad, dg are equal (i. 47) to the square of ag; therefore the squares of ad, dg are double of the squares of ac, cd: but dg is equal to db; therefore the squares of ad, db are double of the squares of ac, cd. Wherefore, if a straight line, &c. Q. E. D.

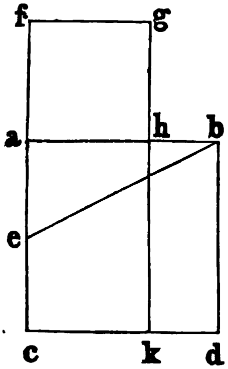
Proposition XI.—Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Let ab be the given straight line; it is required to divide it into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square of the other part.

Upon ab describe (i. 46) the square abdc; bisect (i. 10) ac in e, and join be; produce ca to f, and make (i. 3) ef equal to eb, and upon af describe (i. 46) the square fgha; ab is divided in h, so that the rectangle ab, bh, is equal to the square of ah.

Produce gh to k; because the straight line ac is bisected in e, and produced to the point f, the rectangle cf, fa, together with the square of ae, is equal (ii. 6) to the square of ef: but ef is equal to eb; therefore the rectangle cf, fa, together with the square of ae, is equal to the square of eb: and the squares of ba, ae are equal (i. 47) to the square of **e b**, because the angle **e a b** is a right angle; therefore the rectangle **c f**, **f a**, together with the square of **a e**, is equal to the squares of **b a**, **a e**: take away the square of **a e**, which is common to both, therefore the remaining rectangle **c f**, **f a**, is equal to the square of **a b**; and the figure **f k** is the rectangle contained by **c f**, **f a**, for **a f** is equal to **f g**; and **a d** is the square of **a b**; therefore **f k** is equal to **a d**: take away the common part **a k**, and the remainder **f h** is equal to the remainder **h d**: and **h d** is the rectangle contained by **a b**, **b h**, for **a b** is equal to **b d**; and **f h** is the square of **a h**. Therefore the rectangle **a b**, **b h** is equal to the square of **a h**: wherefore the straight line **a b** is divided in **h**, so that the rectangle **a b**, **b h**, is equal to the square of **a h**. Which was to be done.

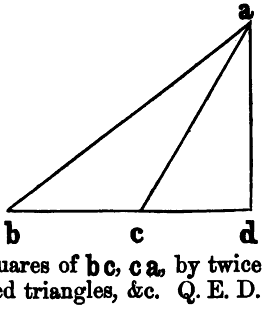


Proposition XII.—Theorem.

In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let **a b c** be an obtuse-angled triangle, having the obtuse angle **a c b**, and from the point **a** let **a d** be drawn (i. 12) perpendicular to **b c** produced. The square of **a b** is greater than the squares of **a c**, **c b**, by twice the rectangle **b c**, **c d**.

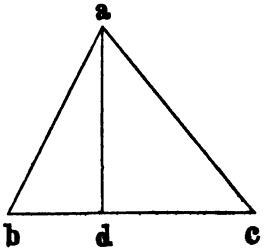
Because the straight line **b d** is divided into two parts in the point **c**, the square of **b d** is equal (ii. 4) to the squares of **b c**, **c d**, and twice the rectangle **b c**, **c d**. To each of these equals add the square of **d a**; and the squares of **b d**, **d a** are equal to the squares of **b c**, **c d**, **d a**, and twice the rectangle **b c**, **c d**. But the square of **b a** is equal (i. 47) to the squares of **b d**, **d a**, because the angle at **d** is a right angle; and the square of **c a** is equal (i. 47) to the squares of **c d**, **d a**. Therefore the square of **b a** is equal to the squares of **b c**, **c a**, and twice the rectangle **b c**, **c d**; that is, the square of **b a** is greater than the squares of **b c**, **c a**, by twice the rectangle **b c**, **c d**. Therefore, in obtuse-angled triangles, &c. Q. E. D.



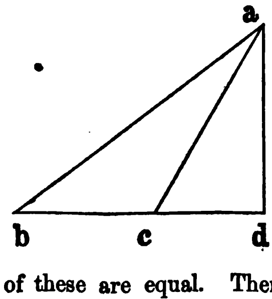
Proposition XIII.—Theorem.

In every triangle, the square of the side subtending any of the acute angles is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.

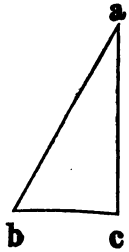
Let **a b c** be any triangle, and the angle at **b** one of its acute angles, and upon **b c**, one of the sides containing it, let fall the perpendicular (i. 12) **a d** from the opposite angle: the square of **a c**, opposite to the angle **b**, is less than the squares of **c b**, **b a**, by twice the rectangle **c b**, **b d**.



First, let **a d** fall within the triangle **a b c**; and because the straight line **c b** is divided into two parts in the point **d**, the squares of **c b**, **b d**, are equal (ii. 7) to twice the rectangle contained by **c b**, **b d**, and the square of **d c**. To each of these equals add the square of **a d**; therefore the squares of **c b**, **b d**, **d a**, are equal to twice the rectangle **c b**, **b d**, and the squares of **a d**, **d c**. But the square of **a b** is equal (i. 47) to the squares of **b d**, **d a**, because the angle **b d a** is a right angle; and the square of **a c** is equal to the squares of **a d**, **d c**. Therefore the squares of **c b**, **b a** are equal to the square of **a c**, and twice the rectangle **c b**, **b d**; that is, the square of **a c** alone is less than the squares of **c b**, **b a** by twice the rectangle **c b**, **b d**.



Secondly, let **a d** fall without the triangle **a b c**. Then, because the angle at **d** is a right angle, the angle **a c b** is greater (i. 16) than a right angle; and therefore the square of **a b** is equal (ii. 12) to the squares of **a c**, **c b**, and twice the rectangle **b c**, **c d**. To these equals add the square of **b c**, and the squares of **a b**, **b c** are equal to the square of **a c**, and twice the square of **b c**, and twice the rectangle **b c**, **c d**. But because **b d** is divided into two parts in **c**, the rectangle **d b**, **b c** is equal (ii. 3) to the rectangle **b c**, **c d** and the square of **b c**. And the doubles of these are equal. Therefore the squares of **a b**, **b c** are equal to the square of **a c**, and twice the rectangle **d b**, **b c**. Therefore the square of **a c** alone is less than the squares of **a b**, **b c**, by twice the rectangle **d b**, **b c**.



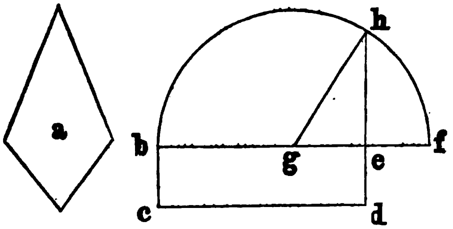
Lastly, let the side **a c** be perpendicular to **b c**; then is **b c** the straight line between the perpendicular and the acute angle at **b**; and it is manifest, that the squares of **a b**, **b c**, are equal (i. 47) to the square of **a c** and twice the square of **b c**. Therefore, in every triangle, &c. Q. E. D.

Proposition XIV.—Problem.

To describe a square that shall be equal to a given rectilineal figure.

Let **a** be the given rectilineal figure; it is required to describe a square that shall be equal to **a**.

Describe (i. 45) the rectangular parallelogram **bcde** equal to the rectilineal figure **a**. If then the sides of it, **be**, **ed**, are equal to one another, it is a square, and what was required is now done. But if they are not equal, produce one of them **be** to **f**, and make **ef** equal to **ed**, and bisect **bf** in **g**; and from the centre **g**, at the distance **gb** or **gf**, describe the semicircle **bhf**, and produce **de** to **h**, and join **gh**: therefore because the straight line **bf** is divided into two equal parts in the point **g**, and into two unequal at **e**, the rectangle **be**, **ef**, together with the square of **eg**, is equal (ii. 5) to the square of **gf**: but **gf** is equal to **gh**: therefore the rectangle **be**, **ef**, together with the square of **eg**, is equal to the square of **gh**: but the squares of **he**, **eg** are equal (i. 47) to the square of **gh**: therefore the rectangle **be**, **ef**, together with the square of **eg**, is equal to the squares of **he**, **eg**; take away the square of **eg**, which is common to both; and the remaining rectangle **be**, **ef** is equal to the square of **eh**: but the rectangle contained by **be**, **ef** is the parallelogram **bd**, because **ef** is equal to **ed**; therefore **bd** is equal to the square of **eh**; but **bd** is equal to the rectilineal figure **a**; therefore the rectilineal figure **a** is equal to the square of **eh**. Wherefore a square has been made equal to the given rectilineal figure **a**, viz. the square described upon **eh**. Which was to be done.



EXERCISES ON BOOK II.

SECT. I.—PROBLEMS.

1. Given the difference between two squares, to find a line whose square shall be equal to that difference.

2. Given two lines, to produce one of them, so that the rectangle contained by the two given lines shall be equal to the square of the part produced.

3. Given a straight line, to describe on it, as the hypotenuse, a right-angled triangle such that the sum of the hypotenuse and the lesser of the other two sides shall be double of the remaining side of the triangle.

4. Given the sum of the squares of any number of lines, to find a line whose square shall be equal to the given sum.

SECT. II.—THEOREMS.

5. If a straight line be drawn from the vertex of an isosceles triangle to any point in the base, the square described on this line, together with the rectangle contained by the segments of the base, is equal to the square described upon either of the equal sides.

6. The difference between the squares of two unequal straight lines is equal to the rectangle contained by their sum and difference.

7. If a straight line be divided into five equal parts, the square of the whole line is equal to the square of the straight line which is made up of four of those parts, together with the square of the straight line which is made up of three of those parts.

8. If a perpendicular be drawn from the right angle of a right-angled triangle to the hypotenuse, the square of the perpendicular is equal to the rectangle contained by the segments of the hypotenuse.

9. If the base of a triangle be bisected, the sum of the squares of the other two sides is equal to twice the square of half the base and twice the square of the line drawn from the point of bisection to the vertical angle.

10. If the middle points of the opposite sides of a quadrilateral figure be joined, the sum of the squares of the joining lines is equal to one-half the sum of the squares of the diagonals of the quadrilateral.

11. If the sides of a triangle be bisected and lines drawn from the points of bisection to the opposite angles, four times the sum of the squares of these lines shall be equal to three times the sum of the squares of the three sides of the triangle.

BOOK III.

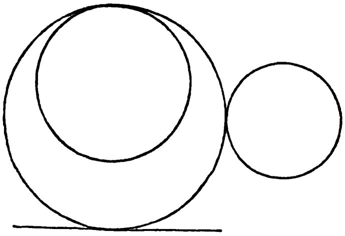
DEFINITIONS.

I. EQUAL circles are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.

This is not a definition, but a theorem, the truth of which is evident; for, if the circles be applied to one another, so that their centres coincide, the circles must likewise coincide, since the straight lines from the centres are equal.

II. A straight line is said to touch a circle when it meets the circle, and being produced does not cut it.

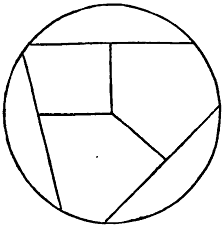
III. Circles are said to touch one another, which meet but do not cut one another.



Definitions II. and III.

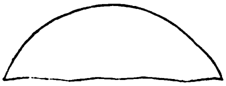
IV. Straight lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

V. And the straight line on which the greater perpendicular falls is said to be farther from the centre.



Definitions IV. and V.

VI. A segment of a circle is the figure contained by a straight line and the circumference it cuts off.

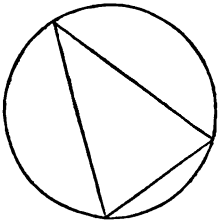


Definition VI.

VII. The angle of a segment is that which is contained by the straight line and the circumference.

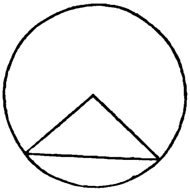
VIII. An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

IX. And an angle is said to insist or stand upon the circumference intercepted between the straight lines that contain the angle.



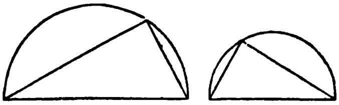
Definitions VIII. and IX.

X. The sector of a circle is the figure contained by two straight lines drawn from the centre, and the circumference between them.



Definition X.

XI. Similar segments of a circle are those in which the angles are equal, or which contain equal angles.



Definition XI.

Proposition I.—Problem.

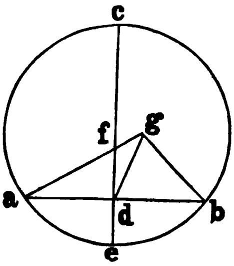
To find the centre of a given circle.

Let **a b c** be the given circle; it is required to find its centre.

Draw within it any straight line **a b**, and bisect (i. 10) it in **d**; from the point **d** draw (i. 11) **d c** at right angles to **a b**, and produce it to **e**, and bisect **c e** in **f**: the point **f** is the centre of the circle **a b c**.

For, if it be not, let, if possible, **g** be the centre, and join **g a**, **g d**,

g b. Then, because **d a** is equal to **d b**, and **d g** common to the two triangles **a d g**, **b d g**, the two sides **a d**, **d g**, are equal to the two **b d**, **d g**, each to each; and the base **g a** is equal to the base **g b**, because they are drawn from the centre **g** *: therefore the angle **a d g** is equal (i. 8) to the angle **g d b**. But when a straight line standing upon another straight line makes the adjacent angles equal to one another, each of the angles is a right angle (i. *def.* 10): therefore the angle **g d b** is a right angle: but **f d b** is likewise a right angle: wherefore the angle **f d b** is equal to the angle **g d b**, the greater to the less, which is impossible: therefore **g** is not the centre of the circle **a b c**. In the same manner it can be shewn, that no other point but **f** is the centre; that is, **f** is the centre of the circle **a b c**. Which was to be found.



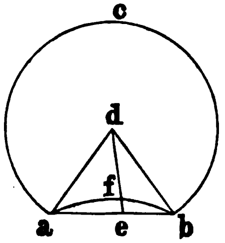
Cor. From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

Proposition II.—Theorem.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let **a b c** be a circle, and **a**, **b** any two points in the circumference; the straight line drawn from **a** to **b** shall fall within the circle.

For, if it do not, let it fall, if possible, without, as **a e b**; find (i. 3) **d** the centre of the circle **a b c**; and join **a d**, **d b**, and produce **d f**, any straight line meeting the circumference **a b**, to **e**: then because **d a** is equal to **d b**, the angle **d a b** is equal (i. 5) to the angle **d b a**; and because **a e**, a side of the triangle **d a e**, is produced to **b**, the angle **d e b** is greater (i. 16) than the angle **d a e**; but **d a e** is equal to the angle **d b e**; therefore the angle **d e b** is greater than the angle **d b e**: but to the greater angle the greater side is opposite (i. 19); **d b** is therefore greater than **d e**: but **d b** is equal to **d f**; wherefore **d f** is greater than **d e**, the less than the greater, which is impossible: therefore the straight line drawn from **a** to **b** does not fall without the circle. In the same manner, it may be demonstrated that it does not fall upon the circumference; it falls therefore within it. Wherefore, if any two points, &c. Q. E. D.



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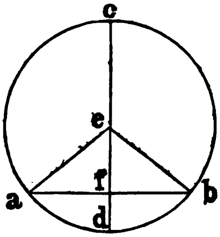
* Whenever the expression "straight lines from the centre," or "drawn from the centre," occurs, it is to be understood that they are drawn to the circumference.

PROPOSITION III.—THEOREM.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and if it cuts it at right angles, it shall bisect it.

LET **abc** be a circle; and let **cd**, a straight line drawn through the centre, bisect any straight line **ab**, which does not pass through the centre, in the point **f**; it cuts it also at right angles.

Take (i. 3) **e** the centre of the circle, and join **ea**, **eb**. Then, because **af** is equal to **fb**, and **fe** common to the two triangles **afe**, **bfe**, there are two sides in the one equal to two sides in the other, and the base **ea** is equal to the base **eb**; therefore the angle **afe** is equal (i. 8) to the angle **bfe**: but when a straight line standing upon another makes the adjacent angles equal to one another, each of them is a right angle (i. *def.* 10): therefore each of the angles **afe**, **bfe**, is a right angle; wherefore the straight line **cd**, drawn through the centre, bisecting another **ab** that does not pass through the centre, cuts the same at right angles.



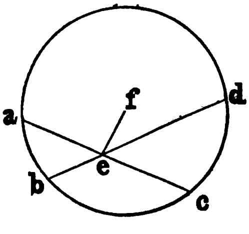
But let **cd** cut **ab** at right angles; **cd** also bisects it; that is, **af** is equal to **fb**.

The same construction being made, because **ea**, **eb**, from the centre are equal to one another, the angle **eaf** is equal (i. 5) to the angle **ebf**: and the right angle **afe** is equal to the right angle **bfe**: therefore, in the two triangles **eaf**, **ebf**, there are two angles in one equal to two angles in the other, and the side **ef**, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (i. 26); **af** therefore is equal to **fb**. Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION IV.—THEOREM.

If in a circle two straight lines cut one another which do not both pass through the centre, they do not bisect each other.

LET **abcd** be a circle, and **ac**, **bd** two straight lines in it which cut one another in the point **e**, and do not both pass through the centre: **ac**, **bd**, do not bisect one another.



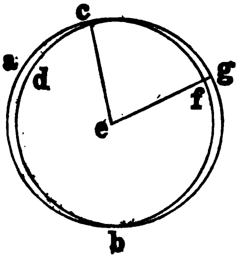
For, if it be possible, let **ae** be equal to **ec**, and **be** to **ed**: if one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre: but if neither of them pass through the centre, take (i. 3) **f** the centre of the circle, and join **ef**; and because **fe**, a straight line through the centre, bisects another **ac** which does not pass through the centre, it shall cut it at right (iii. 3) angles: wherefore **fea** is a right angle.

Again, because the straight line **fe** bisects the straight line **bd**, which does not pass through the centre, it shall cut it at right angles (iii. 3): wherefore **feb** is a right angle: and **fea** was shewn to be a right angle; therefore **fea** is equal to the angle **feb**, the less to the greater, which is impossible: therefore **ac**, **bd** do not bisect one another. Wherefore, if in a circle, &c. Q. E. D.

PROPOSITION V.—THEOREM.

If two circles cut one another, they shall not have the same centre.

LET the two circles **abc**, **cdg** cut one another in the points **b**, **c**; they have not the same centre.

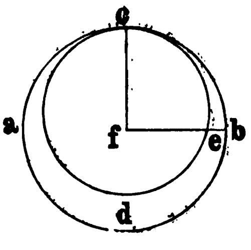


For, if it be possible, let **e** be their centre; join **ec**, and draw any straight line **efg** meeting them in **f** and **g**; and because **e** is the centre of the circle **abc**, **ce** is equal to **ef**. Again, because **e** is the centre of the circle **cdg**, **ce** is equal to **eg**: but **ce** was shewn to be equal to **ef**: therefore **ef** is equal to **eg**, the less to the greater, which is impossible: therefore **e** is not the centre of the circles **abc**, **cdg**. Wherefore, if two circles, &c. Q. E. D.

PROPOSITION VI.—THEOREM.

If two circles touch one another internally, they shall not have the same centre.

LET the two circles **abc**, **cde**, touch one another internally in the point **c**; they have not the same centre.

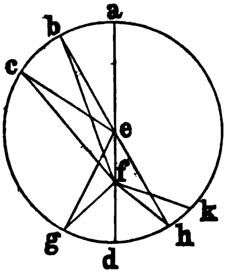


For, if they have, let it be **f**; join **fc** and draw any straight line **feb** meeting them in **e** and **b**; and because **f** is the centre of the circle **abc**, **cf** is equal to **fb**; also, because **f** is the centre of the circle **cde**, **cf** is equal to **fe**: and **cf** was shewn to be equal to **fb**; therefore **fe** is equal to **fb**, the less to the greater, which is impossible: wherefore **f** is not the centre of the circles **abc**, **cde**. Therefore, if two circles, &c. Q. E. D.

PROPOSITION VII.—THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and of any others, that which is nearer to the line which *passes through the centre is always greater than one more remote. And from the same point there can be drawn only two straight lines that are equal to one another, one upon each side of the shortest line.*

LET **abcd** be a circle, and **ad** its diameter, in which let any point **f** be taken which is not the centre. Let the centre be **e**; of all the straight lines **fb**, **fc**, **fg**, &c. that can be drawn from **f** to the circumference, **fa** is the greatest, and **fd**, the other part of the diameter **ad**, is the least: and of the others, **fb** is greater than **fc**, and **fc** than **fg**.



Join **be**, **ce**, **ge**; and because two sides of a triangle are greater (i. 20) than the third, **be**, **ef**, are greater than **bf**; but **ae** is equal to **eb**; therefore **ae**, **ef**, that is **af**, is greater than **bf**. Again, because **be** is equal to **ce**, and **fe** common to the triangles **bef**, **cef**, the two sides **be**, **ef** are equal to the two **ce**, **ef**; but the angle **bef** is greater than the angle **cef**; therefore the base **bf** is greater (i. 24) than the base **fc**: for the same reason **cf** is greater than **gf**. Again, because **gf**, **fe** are greater (i. 20) than **eg**, and **eg** is equal to **ed**, **gf**, **fe** are greater than **ed**; take away the common part **fe**, and the remainder **gf** is greater than the remainder **fd**: therefore **fa** is the greatest, and **fd** the least of all the straight lines from **f** to the circumference; and **bf** is greater than **cf**, and **cf** than **gf**.

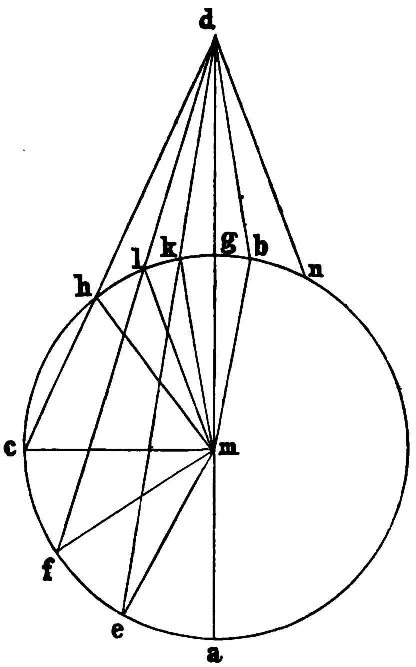
Also there can be drawn only two equal straight lines from the point **f** to the circumference, one upon each side of the shortest line **fd**: at the point **e** in the straight line **ef**, make (i. 23) the angle **feh** equal to the angle **gef**, and join **fh**: then, because **ge** is equal to **eh**, and **ef** common to the two triangles **gef**, **hef**; the two sides **ge**, **ef** are equal to the two **he**, **ef**; and the angle **gef** is equal to the angle **hef**; therefore the base **fg** is equal (i. 4) to the base **fh**. But besides **fh**, no other straight line can be drawn from **f** to the circumference equal to **fg**: for, if there can, let it be **fk**; and because **fk** is equal to **fg**, and **fg** to **fh**, **fk** is equal to **fh**; that is, a line nearer to that which passes through the centre, is equal to one which is more remote; which is impossible. Therefore, if any point be taken, &c. Q. E. D.

PROPOSITION VIII.—THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than the more remote: and only two equal straight lines can be drawn from the point unto the circumference, one upon each side of the least.

LET **abc** be a circle, and **d** any point without it, from which let the straight lines **da**, **de**, **df**, **dc** be drawn to the circumference, whereof **da** passes through the centre. Of those which fall upon the concave part of the circumference **aefc**, the greatest is **ad** which passes through the centre; and the nearer to it is always greater than the more remote, viz., **de** than **df**, and **df** than **dc**; but of those which fall upon the convex circumference **hlkg**, the least is **dg** between the point **d** and the diameter **ag**; and the nearer to it is always less than the more remote, viz. **dk** than **dl**, and **dl** than **dh**.

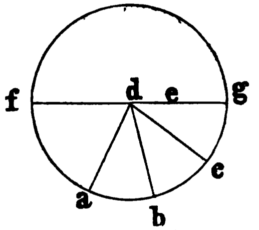
Take (i. 3) **m** the centre of the circle **abc**, and join **me**, **mf**, **mc**, **mk**, **ml**, **mh**. And because **am** is equal to **me**, add **md** to each, therefore **ad** is equal to **em**, **md**; but **em**, **md** are greater (i. 20) than **ed**; therefore also **ad** is greater than **ed**. Again, because **me** is equal to **mf**, and **md** common to the triangles **emd**, **fmd**; **em**, **md** are equal to **fm**, **md**: but the angle **emd** is greater than the angle **fmd**; therefore the base **ed** is greater than the base **fd** (i. 24). In like manner it may be shewn that **fd** is greater than **cd**. Therefore **da** is the greatest, and **de** greater than **df**, and **df** than **dc**. And because **mk**, **kd** are greater (i. 20) than **md**, and **mk** is equal to **mg**, the remainder **kd** is greater (4 *ax.*) than the remainder **gd**, that is, **gd** is less than **kd**. And because **mk**, **dk** are drawn to the point **k** within the triangle **mld** from **m**, **d**, the extremities of its side **md**; **mk**, **kd**, are less (i. 21) than **ml**, **ld**, whereof **mk** is equal to **ml**; therefore the remainder **dk** is less than the remainder **dl**. In like manner it may be shewn that **dl** is less than **dh**: therefore **dg** is the least, and **dk** less than **dl**, and **dl** than **dh**. Also there can be drawn only two equal straight lines from the point **d** to the circumference, one upon each side of the least. At the point **m**, in the straight line **md**, make the angle **dmb** equal to the angle **dmk**, and join **db**. And because **mk** is equal to **mb**, and **md** common to the triangles **kmd**, **bmd**, the two sides **km**, **md** are equal to the two **bm**, **md**, and the angle **kmd** is equal to the angle **bmd**; therefore the base **dk** is equal (i. 4) to the base **db**. But besides **db** there can be no straight line drawn from **d** to the circumference equal to **dk**. For if there can let it be **dn**; and because **dk** is equal to **dn** and also to **db**, therefore **db** is equal to **dn**; that is, nearer to the least equal to the more remote, which is impossible: therefore any point, &c. Q. E. D.



PROPOSITION IX.—THEOREM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

LET the point **d** be taken within the circle **a b c**, from which to the circumference there fall more than two equal straight lines, viz. **d a**, **d b**, **d c**, the point **d** is the centre of the circle.

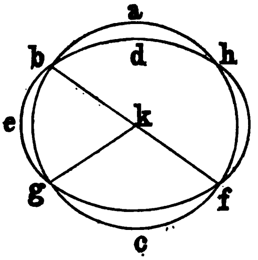


For, if not, let **e** be the centre, join **d e** and produce it to the circumference in **f**, **g**: then **f g** is a diameter of the circle **a b c**. And because in **f g**, the diameter of the circle **a b c**, there is taken the point **d**, which is not the centre, **d g** shall be the greatest line from it to the circumference, and **d c** greater (iii. 7) than **d b**, and **d b** than **d a**. But they are likewise equal, which is impossible; therefore **e** is not the centre of the circle **a b c**. In like manner it may be demonstrated that no other point but **d** is the centre; **d** therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.

PROPOSITION X.—THEOREM.

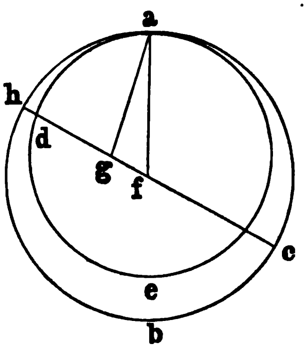
One circumference of a circle cannot cut another in more than two points.

IF it be possible, let the circumference **a b c** cut the circumference **d e f** in more than two points, viz. in **b**, **g**, **f**; take the centre **k** of the circle **a b c**, and join **k b**, **k g**, **k f**. And because within the circle **d e f** there is taken the point **k**, from which to the circumference **d e f** fall more than two equal straight lines **k b**, **k g**, **k f**, the point **k** is the centre of the circle **d e f** (iii. 9). But **k** is also the centre of the circle **a b c**; therefore the same point is the centre of two circles that cut one another, which is impossible (iii. 5). Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.



PROPOSITION XI.—THEOREM.

If two circles touch each other internally, the straight line which joins their centres being produced shall pass through the point of contact.



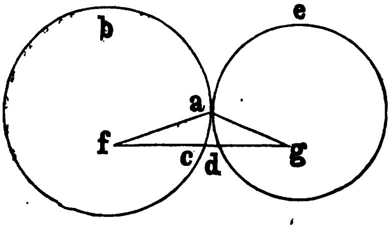
LET the two circles **a b c**, **a d e** touch each other internally in the point **a**, and let **f** be the centre of the circle **a b c**, and **g** the centre of the circle **a d e**. The straight line which joins the centres **f**, **g**, being produced, passes through the point **a**.

For, if not, let it fall otherwise, if possible, as **f g d h**, and join **a f**, **a g**. And because **a g**, **g f** are greater (i. 20) than **f a**, that is, than **f h**, for **f a** is equal to **f h**, both being from the same centre; take away the common part **f g**; therefore the remainder **a g** is greater than the remainder **g h**. But **a g** is equal to **g d**; therefore **g d** is greater than **g h**, the less than the greater, which is impossible. Therefore the straight line which joins the points **f**, **g** cannot fall otherwise than upon the point **a**, that is, it must pass through it. Therefore, if two circles, &c. Q. E. D.

PROPOSITION XII.—THEOREM.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

LET the two circles **a b c**, **a d e** touch each other externally in the point **a**; and let **f** be the centre of the circle **a b c**, and **g** the centre of **a d e**. The straight line which joins the points **f**, **g** shall pass through the point of contact **a**.

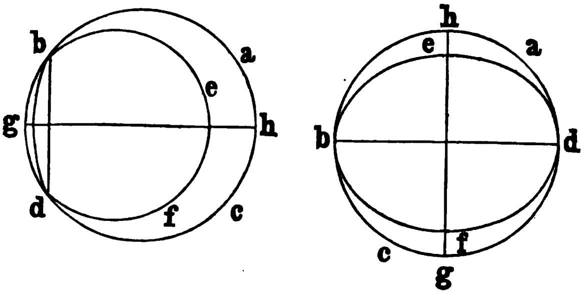


For, if not, let it pass otherwise, if possible, as **f c d g**, and join **f a**, **a g**. And because **f** is the centre of the circle **a b c**, **a f** is equal to **f c**. Also, because **g** is the centre of the circle **a d e**, **a g** is equal to **g d**. Therefore **f a**, **a g** are equal to **f c**, **d g**; wherefore the whole **f g** is greater than **f a**, **a g**. But it is also less (i. 20); which is impossible. Therefore the straight line which joins the points **f**, **g** shall not pass otherwise than through the point of contact **a**, that is, it must pass through it. Therefore, if two circles, &c. Q. E. D.

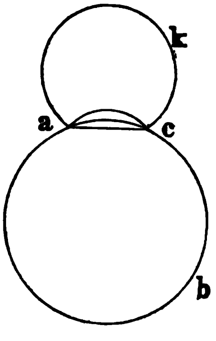
PROPOSITION XIII.—THEOREM.

One circle cannot touch another in more points than one, whether it touches it on the inside or outside.

For, if it be possible, let the circle **e b f** touch the circle **a b c** in more points than one, and first on the inside, in the points **b**, **d**; join **b d**, and draw (i. 10, 11) **g h** bisecting **b d** at right angles. Therefore, because the points **b**, **d** are in the circumference of each of the circles, the straight line **b d** falls within each of them (iii. 2): and their centres are in the straight line **g h** which bisects **b d** at right angles (iii. 1. *Cor.*). Therefore **g h** passes through the point of contact (iii. 11); but it does not pass through it, because the points **b**, **d** are without the straight line **g h**, which is absurd. Therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another on the outside in more than one point; for, if it be possible, let the circle **a c k** touch the circle **a b c** in the points **a**, **c**, and join **a c**. Therefore, because the two points **a**, **c**, are in the circumference of the circle **a c k**, the straight line **a c** which joins them shall fall within the circle **a c k** (iii. 2). And the circle **a c k** is without the circle **a b c**; and therefore the straight line **a c** is without this last circle; but because the points **a**, **c** are in the circumference of the circle **a b c**, the straight line **a c** must be within (iii. 2) the same circle, which is absurd. Therefore one circle cannot touch another on the outside in more than one point. And it has been shewn that they cannot touch on the inside in more points than one. Therefore, one circle, &c. Q. E. D.

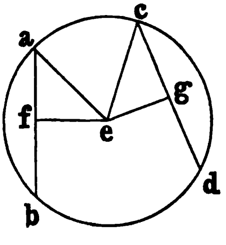


PROPOSITION XIV.—THEOREM.

Equal straight lines in a circle are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

LET the straight lines **a b**, **c d**, in the circle **a b d c**, be equal to one another, they are equally distant from the centre.

Take **e** the centre of the circle **a b d c**, and from it draw **e f**, **e g**, perpendiculars to **a b**, **c d**. Then, because the straight line **e f**, passing through the centre, cuts the straight line **a b**, which does not pass through the centre, at right angles, it also bisects it (iii. 3); wherefore **a f** is equal to **f b**, and **a b** double of **a f**. For the same reason, **c d** is double of **c g**; and **a b** is equal to **c d**; therefore **a f** is equal to **c g**. And because **a e** is equal to **e c**, the square of **a e** is equal to the square of **e c**; but the squares of **a f**, **f e**, are equal (i. 47) to the square of **a e**, because the angle **a f e** is a right angle; and, for the like reason, the squares of **e g**, **g c**, are equal to the square of **e c**; therefore the squares of **a f**, **f e**, are equal to the squares of **c g**, **g e**, of which the square of **a f** is equal to the square of **c g**, because **a f** is equal to **c g**; therefore the remaining square of **f e** is equal to the remaining square of **e g**, and the straight line **e f** is therefore equal to **e g**. But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (iii. *def.* 4). Therefore **a b**, **c d** must be equally distant from the centre.



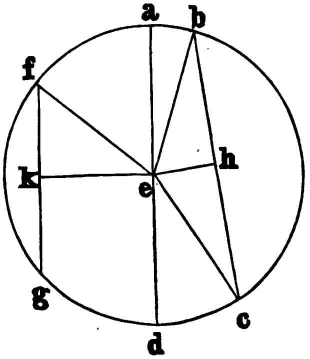
Next, if the straight lines **a b**, **c d** be equally distant from the centre, that is, if **f e** be equal to **e g**; **a b** is equal to **c d**. For, the same construction being made, it may, as before, be demonstrated, that **a b** is double of **a f**, and **c d** double of **c g**, and that the squares of **e f**, **f a** are equal to the squares of **e g**, **g c**; of which the square of **f e** is equal to the square of **e g**, because **f e** is equal to **e g**; therefore the remaining square of **a f** is equal to the remaining square of **c g**; and the straight line **a f** is therefore equal to **c g**: and **a b** is double of **a f**, and **c d** double of **c g**; wherefore **a b** is equal to **c d**. Therefore equal straight lines, &c. Q. E. D.

PROPOSITION XV.—THEOREM.

The diameter is the greatest straight line in a circle, and of all others that which is nearer to the centre is always greater than one more remote, and the greater is nearer to the centre than the less.

LET **a b c d** be a circle, of which the diameter is **a d**, and the centre **e**; and let **b c** be nearer to the centre than **f g**; **a d** is greater than any straight line **b c**, which is not a diameter, and **b c** greater than **f g**.

From the centre **e**, draw **e h**, **e k**, perpendiculars to **b c**, **f g**, and join **e b**, **e c**, **e f**; and because **a e** is equal to **e b**, and **e d** to **e c**, **a d** is equal to **e b**, **e c**; but **e b**, **e c** are greater than **b c** (i. 20); wherefore, also, **a d** is greater than **b c**.



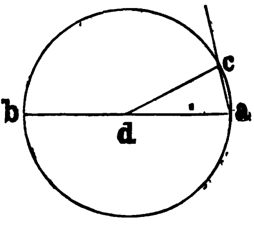
And, because **b c** is nearer to the centre than **f g**, **e h** is less (iii. *def.* 5) than **e k**. But, as was demonstrated in the preceding, **b c** is double of **b h**, and **f g** double of **f k**, and the squares of **e h**, **h b** are equal to the squares of **e k**, **k f**, of which the square of **e h** is less than the square of **e k**, because **e h** is less than **e k**; therefore the square of **b h** is greater than the square of **f k**, and the straight line **b h** greater than **f k**, and therefore **b c** is greater than **f g**.

Next, let **b c** be greater than **f g**; **b c** is nearer to the centre than **f g**, that is, the same construction being made, **e h** is less than **e k**. Because **b c** is greater than **f g**, **b h** likewise is greater than **f k**; and the squares of **b h**, **h e** are equal to the squares of **f k**, **k e**, of which the square of **b h** is greater than the square of **f k**, because **b h** is greater than **f k**; therefore the square of **e h** is less than the square of **e k**, and the straight line **e h** less than **e k**. Wherefore the diameter, &c. Q. E. D.

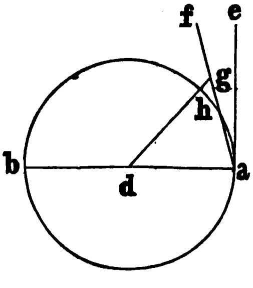
PROPOSITION XVI.—THEOREM.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn between that straight line and the circumference from the extremity, so as not to cut the circle; or, which is the same thing, no straight line can make so great an acute angle with the diameter at its extremity, or so small an angle with the straight line which is at right angles to it, as not to cut the circle.

LET **a b c** be a circle, the centre of which is **d**, and the diameter **a b**: the straight line drawn at right angles to **a b** from its extremity **a**, shall fall without the circle.



For, if it does not, let it fall, if possible, within the circle, as **a c**, and draw **d c** to the point **c** where it meets the circumference. And because **d a** is equal to **d c**, the angle **d a c** is equal (i. 5) to the angle **a c d**; but **d a c** is a right angle, therefore **a c d** is a right angle, and the angles **d a c**, **a c d** are therefore equal to two right angles, which is impossible (i. 17). Therefore the straight line drawn from **a** at right angles to **b a**, does not fall within the circle. In the same manner it may be demonstrated, that it does not fall upon the circumference; therefore it must fall without the circle as **a e**.



And between the straight line **a e** and the circumference, no straight line can be drawn from the point **a** which does not cut the circle. For, if possible, let **f a** be between them, and from the point **d** draw (i. 12) **d g** perpendicular to **f a**, and let it meet the circumference in **h**; and because **a g d** is a right angle, and **d a g** less (i. 17) than a right angle, **d a** is greater (i. 19) than **d g**. But **d a** is equal to **d h**; therefore **d h** is greater than **d g**, the less than the greater, which is impossible. Therefore no straight line can be drawn from the point **a** between **a e** and the circumference, which does not cut the circle; or, which amounts to the same thing, however great an acute angle a straight line makes with the diameter at the point **a**, or however small an angle it makes with **a e**, the circumference passes between that straight line and the perpendicular **a e**. And this is all that is to be understood, when, in the Greek text, and translations from it, the angle of the semicircle is said to be greater than any acute rectilineal angle, and the remaining angle less than any rectilineal angle. Q. E. D.

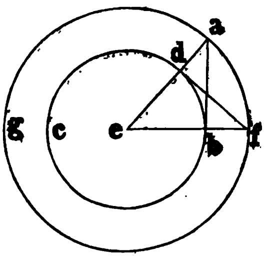
Cor. From this it is manifest, that the straight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because if it did meet the circle in two, it would fall within it (iii. 2). Also, it is evident that there can be but one straight line which touches the circle in the same point.

Proposition XVII.—Problem.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

First, let **a** be a given point without the given circle **b c d**; it is required to draw a straight line from **a**, which shall touch the circle.

Find (iii. 1) the centre **e** of the circle, and join **a e**; and from the centre **e**, at the distance **e a**, describe the circle **a f g**; from the point **d** draw (i. 11) **d f** at right angles to **e a**, and join **e b f**, **a b**: **a b** touches the circle **b c d**.



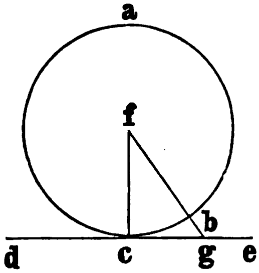
Because **e** is the centre of the circles **b c d**, **a f g**; **e a** is equal to **e f**, and **e d** to **e b**; therefore the two sides **a e**, **e b** are equal to the two **f e**, **e d**, and they contain the angle at **e** common to the two triangles **a e b**, **f e d**: therefore the base **d f** is equal to the base **a b**; and the triangle **f e d** to the triangle **a e b**, and the other angles to the other angles (i. 4): therefore the angle **e b a** is equal to the angle **e d f**; but **e d f** is a right angle, wherefore **e b a** is a right angle; and **e b** is drawn from the centre: but a straight line drawn from the extremity of a diameter, at right angles to it, touches the circle (iii. 16. *cor.*): therefore **a b** touches the circle; and it is drawn from the given point **a**. Which was to be done.

But if the given point be in the circumference of the circle, as the point **d**, draw **d e** to the centre **e**, and **d f** at right angles to **d e**; **d f** touches the circle (iii. 16. *cor.*).

PROPOSITION XVIII.—THEOREM.

If a straight line touches a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

LET the straight line **d e** touch the circle **a b c** in the point **c**; take the centre **f**, and draw the straight line **f c**; **f c** is perpendicular to **d e**.

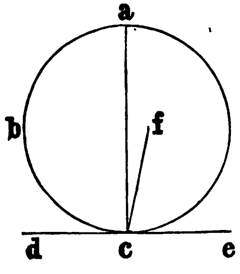


For, if it be not, from the point **f** draw **f b g** perpendicular to **d e**; and because **f g c** is a right angle, **g c f** (i. 17) is an acute angle; and to the greater angle the greatest (i. 19) side is opposite: therefore **f c** is greater than **f g**; but **f c** is equal to **f b**; therefore **f b** is greater than **f g**, the less than the greater, which is impossible: wherefore **f g** is not perpendicular to **d e**. In the same manner it may be shewn, that no other is perpendicular to it besides **f c**, that is, **f c** is perpendicular to **d e**. Therefore, if a straight line, &c. Q. E. D.

PROPOSITION XIX.—THEOREM.

If a straight line touches a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

LET the straight line **d e** touch the circle **a b c** in **c**; and from **c** let **c a** be drawn at right angles to **d e**; the centre of the circle is in **c a**.

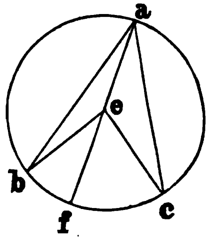


For, if not, let **f** be the centre, if possible, and join **c f**; because **d e** touches the circle **a b c**, and **f c** is drawn from the centre to the point of contact, **f c** is perpendicular (iii. 18) to **d e**; therefore **f c e** is a right angle. But **a c e** is also a right angle; therefore the angle **f c e** is equal to the angle **a c e**, the less to the greater, which is impossible: wherefore **f** is not the centre of the circle **a b c**. In the same manner, it may be shewn, that no other point which is not in **c a**, is the centre; that is, the centre is in **c a**. Therefore, if a straight line, &c. Q. E. D.

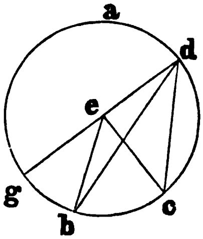
Proposition XX.—Theorem.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

Let **abc** be a circle, and **bec** an angle at the centre, and **bac** an angle at the circumference, which have the same circumference **bc** for their base; the angle **bec** is double of the angle **bac**.



First, let **e**, the centre of the circle, be within the angle **bac**, and join **ae**, and produce it to **f**: because **ea** is equal to **eb**, the angle **eab** is equal (i. 5) to the angle **eba**; therefore the angles **eab**, **eba** are double of the angle **eab**; but the angle **bef** is equal (i. 32) to the angles **eab**, **eba**; therefore also the angle **bef** is double of the angle **eab**. For the same reason, the angle **fec** is double of the angle **eac**: therefore the whole angle **bec** is double of the whole angle **bac**.

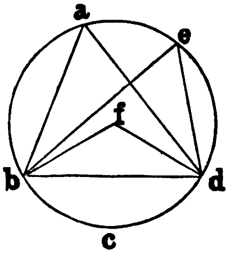


Again, let **e**, the centre of the circle, be without the angle **bdc**, and join **de**, and produce it to **g**, it may be demonstrated, as in the first case, that the angle **gec** is double of the angle **gdc**, and that **geb**, a part of the first, is double of **gdb**, a part of the other; therefore the remaining angle **bec** is double of the remaining angle **bdc**. Therefore the angle at the centre, &c. Q. E. D.

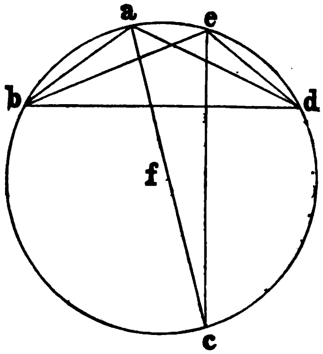
Proposition XXI.—Theorem.

The angles in the same segment of a circle are equal to one another.

Let **abcd** be a circle, and **bad**, **bed** angles in the same segment **baed**: the angles **bad**, **bed** are equal to one another.



Take **f** the centre of the circle **abcd**: and, first, let the segment **baed** be greater than a semicircle, and join **bf**, **fd**: and because the angle **bfd** is at the centre, and the angle **bad** at the circumference, and that they have the same part of the circumference, viz. **bcd** for their base; therefore the angle **bfd** is double (iii. 20) of the angle **bad**. For the same reason, the angle **bfd** is double of the angle **bed**: therefore the angle **bad** is equal to the angle **bed**.



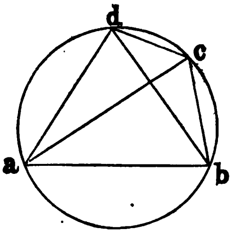
But if the segment **b a e d** be not greater than a semicircle, let **b a d**, **b e d** be angles in it; these also are equal to one another: draw **a f** to the centre, and produce it to **c**, and join **c e**: therefore the segment **b a d c** is greater than a semicircle; and the angles in it **b a c**, **b e c** are equal, by the first case: for the same reason, because **c b e d** is greater than a semicircle, the angles **c a d**, **c e d**, are equal: therefore the whole angle **b a d** is equal to the whole angle **b e d**. Wherefore the angles in the same segment, &c. Q. E. D.

Proposition XXII.—Theorem.

The opposite angles of any quadrilateral figure described in a circle are together equal to two right angles.

Let **a b c d** be a quadrilateral figure in the circle **a b c d**; any two of its opposite angles are together equal to two right angles.

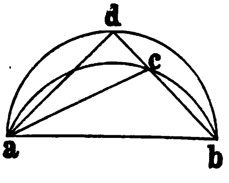
Join **a c**, **b d**; and because the three angles of every triangle are equal (i. 32) to two right angles, the three angles of the triangle **c a b**, viz. the angles **c a b**, **a b c**, **b c a**, are equal to two right angles: but the angle **c a b** is equal (iii. 21) to the angle **c d b**, because they are in the same segment **b a d c**, and the angle **a c b** is equal to the angle **a d b**, because they are in the same segment **a d c b**: therefore the whole angle **a d c** is equal to the angles **c a b**, **a c b**: to each of these equals add the angle **a b c**; therefore the angles **a b c**, **c a b**, **b c a** are equal to the angles **a b c**, **a d c**. But **a b c**, **c a b**, **b c a**, are equal to two right angles; therefore also the angles **a b c**, **a d c**, are equal to two right angles. In the same manner, the angles **b a d**, **d c b**, may be shewn to be equal to two right angles. Therefore the opposite angles, &c. Q. E. D.



Proposition XXIII.—Theorem.

Upon the same straight line and upon the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If it be possible, let the two similar segments of circles, viz. **a c b**, **a d b** be upon the same side of the same straight line **a b**, not coinciding with one another. Then because the circle **a c b** cuts the circle **a d b** in the two points **a**, **b** they cannot cut one another in any other point (iii. 10). One of the segments must therefore fall within the other.



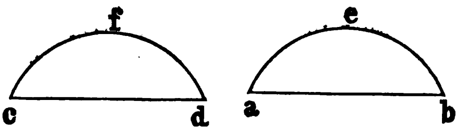
Let **acb** fall within **adb**, and draw the straight line **bcd**, and join **ca**, **da**. And because the segment **acb** is similar to the segment **adb**, and that similar segments of circles contain (iii. *def.* 11) equal angles; the angle **acb** is equal to the angle **adb**, the exterior to the interior, which is impossible (i. 16). Therefore there cannot be two similar segments of a circle upon the same side of the same line, which do not coincide. Q. E. D.

PROPOSITION XXIV.—THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

Let **aeb**, **cfd** be similar segments of circles upon the equal straight lines **ab**, **cd**; the segment **aeb** is equal to the segment **cfd**.

For if the segment **aeb** be applied to the segment **cfd**, so as the point **a** be on **c**, and the straight line **ab** upon **cd**, the point **b** shall coincide with the point **d**, because **ab** is equal to **cd**. Therefore the straight line **ab** coinciding with **cd**, the segment **aeb** must (iii. 23) coincide with the segment **cfd**, and therefore is equal to it. Wherefore similar segments, &c. Q. E. D.

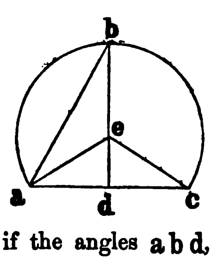


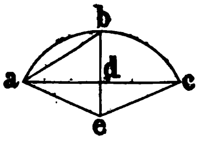
PROPOSITION XXV.—PROBLEM.

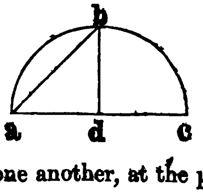
A segment of a circle being given, to describe the circle of which it is the segment.

Let **abc** be the given segment of a circle; it is required to describe the circle of which it is the segment.

Bisect (i. 10) **ac** in **d**, and from the point **d** draw (i. 11) **db** at right angles to **ac**, and join **ab**. First, let the angles **abd**, **bad** be equal to one another; then the straight line **bd** is equal (i. 6) to **da**, and therefore to **dc**; and because the three straight lines **da**, **db**, **dc**, are all equal, **d** is the centre of the circle (iii. 9). From the centre **d**, at the distance of any of the three **da**, **db**, **dc**, describe a circle; this shall pass through the other points; and the circle of which **abc** is a segment is described. And because the centre **d** is in **ac**, the segment **abc** is a semicircle.







But if the angles **abd**, **bad** are not equal to one another, at the point

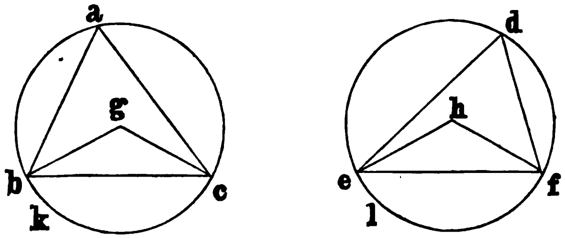
a, in the straight line **a b**, make (i. 23) the angle **b a e** equal to the angle **a b d**, and produce **b d**, if necessary, to **e**, and join **e c**. And because the angle **a b e** is equal to the angle **b a e**, the straight line **b e** is equal (i. 6) to **e a**. And because **a d** is equal to **d c**, and **d e** common to the triangles **a d e**, **c d e**, the two sides **a d**, **d e** are equal to the two **c d**, **d e**, each to each; and the angle **a d e** is equal to the angle **c d e**, for each of them is a right angle; therefore the base **a e** is equal (i. 4) to the base **e c**: but **a e** was shewn to be equal to **e b**, wherefore also **b e** is equal to **e c**. And the three straight lines **a e**, **e b**, **e c** are therefore equal to one another; wherefore (iii. 9) **e** is the centre of the circle. From the centre **e**, at the distance of any of the three **a e**, **e b**, **e c**, describe a circle, this shall pass through the other points; and the circle of which **a b c** is a segment is described. And it is evident that if the angle **a b d** be greater than the angle **b a d**, the centre **e** falls without the segment **a b c**, which therefore is less than a semicircle. But if the angle **a b d** be less than **b a d**, the centre **e** falls within the segment **a b c**, which is therefore greater than a semicircle. Wherefore, a segment of a circle being given, the circle is described of which it is a segment. Which was to be done.

PROPOSITION XXVI.—THEOREM.

In equal circles, equal angles stand upon equal circumferences, whether they be at the centres or circumferences.

LET **a b c**, **d e f** be equal circles, and the equal angles **b g c**, **e h f** at their centres, and **b a c**, **e d f** at their circumferences. The circumference **b k c** is equal to the circumference **e l f**.

Join **b c**, **e f**; and because the circles **a b c**, **d e f**, are equal, the straight lines drawn from their centres are equal. Therefore the two sides **b g**, **g c**, are equal to the two **e h**, **h f**, and the angle at **g** is equal to the angle at **h**; therefore the base **b c** is equal (i. 4) to the base **e f**. And because the angle at **a** is equal to the angle at **d**, the segment **b a c** is similar (iii. *def.* 11) to the segment **e d f**; and they are upon equal straight lines **b c**, **e f**. But similar segments of circles upon equal straight lines are equal (iii. 24) to one another; therefore the segment **b a c** is equal to the segment **e d f**. But the whole circle **a b c** is equal to the whole **d e f**; therefore the remaining segment **b k c** is equal to the remaining segment **e l f**, and the circumference **b k c** to the circumference **e l f**. Wherefore, in equal circles, &c. Q. E. D.

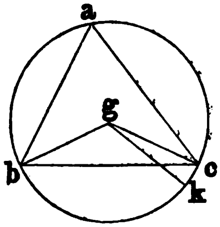


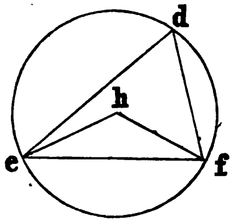
PROPOSITION XXVII.—THEOREM.

In equal circles, the angles which stand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

LET the angles **bgc**, **ehf** at the centres, and **bac**, **edf** at the circumferences of the equal circles **abc**, **def**, stand upon the equal circumferences **bc**, **ef**. The angle **bgc** is equal to the angle **ehf**, and the angle **bac** to the angle **edf**.

If the angle **bgc** be equal to the angle **ehf**, it is manifest (iii. 20) that the angle **bac** is also equal to **edf**. But if not, one of them is the greater. Let **bgc** be the greater, and at the point **g**, in the straight line **bg**, make (i. 23) the angle **bgk** equal to the angle **ehf**; but equal angles stand upon equal circumferences (iii. 26), when they are at the centre; therefore the circumference **bk** is equal to the circumference **ef**. But **ef** is equal to **bc**; therefore also **bk** is equal to **bc**, the less to the greater, which is impossible. Therefore the angle **bgc** is not unequal to the angle **ehf**; that is, it is equal to it. And the angle at **a** is half of the angle **bgc**, and the angle at **d** half of the angle **ehf**: therefore the angle at **a** is equal to the angle at **d**. Wherefore, in equal circles, &c. Q. E. D.



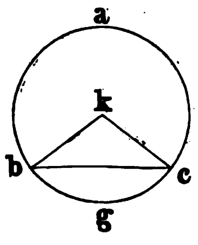


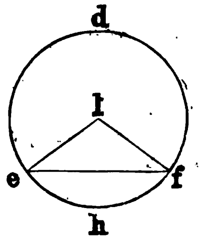
PROPOSITION XXVIII.—THEOREM.

In equal circles, equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less.

LET **abc**, **def** be equal circles, and **bc**, **ef** equal straight lines in them, which cut off the two greater circumferences **bac**, **edf**, and the two less **bgc**, **ehf**; the greater **bac** is equal to the greater **edf**, and the less **bgc** to the less **ehf**.

Take (i. 3) **k**, **l**, the centres of the circles, and join **bk**, **kc**, **el**, **lf**. And because the circles are equal, the straight lines from their centres are equal: therefore **bk**, **kc** are equal to **el**, **lf**; and the base **bc** is equal to the base **ef**; therefore the angle **bkc** is equal (i. 8) to



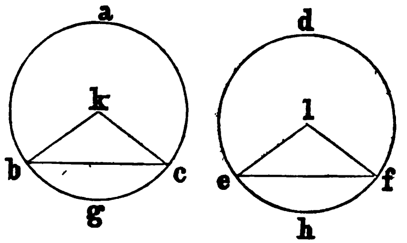


the angle **elf**. But equal angles stand upon equal (iii. 26) circumferences, when they are at the centres; therefore the circumference **bgc** is equal to the circumference **ehf**. But the whole circle **abc** is equal to the whole **edf**; the remaining part therefore of the circumference, viz. **bac**, is equal to the remaining part **edf**. Therefore, in equal circles, &c. Q. E. D.

Proposition XXIX.—Theorem.

In equal circles equal circumferences are subtended by equal straight lines.

Let **abc**, **def** be equal circles, and let the circumferences **bgc**, **ehf** also be equal; and join **bc**, **ef**. The straight line **bc** is equal to the straight line **ef**.



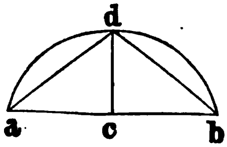
Take (i. 3) **k**, **l**, the centres of the circles, and join **bk**, **kc**, **el**, **lf**. And because the circumference **bgc** is equal to the circumference **ehf**, the angle **bkc** is equal (iii. 27) to the angle **elf**. And because the circles **abc**, **def** are equal, the straight lines from their centres are equal. Therefore **bk**, **kc** are equal to **el**, **lf**, and they contain equal angles: therefore the base **bc** is equal (i. 4) to the base **ef**. Therefore, in equal circles, &c. Q. E. D.

Proposition XXX.—Problem.

To bisect a given circumference; that is, to divide it into two equal parts.

Let **adb** be the given circumference; it is required to bisect it.

Join **ab** and bisect (i. 10) it in **c**; from the point **c** draw **cd** at right angles to **ab**, and join **ad**, **db**. The circumference **adb** is bisected in the point **d**.



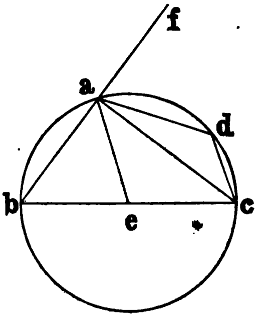
Because **ac** is equal to **cb**, and **cd** common to the triangles **acd**, **bcd**, the two sides **ac**, **cd** are equal to the two **bc**, **cd**; and the angle **acd** is equal to the angle **bcd**, because each of them is a right angle: therefore the base **ad** is equal (i. 4) to the base **bd**. But equal straight lines cut off equal (iii. 28) circumferences, the greater equal to the greater, and the less to the less, and **ad**, **db** are each of them less than a semicircle; because **cd** passes through the centre (i. 3. *cor.*). Wherefore the circumference **ad** is equal to the circumference **db**. Therefore the given circumference is bisected in **d**. Which was to be done.

PROPOSITION XXXI.—THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

LET **abcd** be a circle, of which the diameter is **bc**, and centre **e**; and draw **ca**, dividing the circle into the segments **abc**, **adc**, and join **ba**, **ad**, **dc**; the angle in the semicircle **bac** is a right angle; and the angle in the segment **abc**, which is greater than a semicircle, is less than a right angle; and the angle in the segment **adc**, which is less than a semicircle, is greater than a right angle.

Join **ae**, and produce **ba** to **f**. And because **be** is equal to **ea**, the angle **eab** is equal (i. 5) to **eba**; also, because **ae** is equal to **ec**, the angle **eac** is equal to **eca**; wherefore the whole angle **bac** is equal to the two angles **abc**, **acb**. But **fac**, the exterior angle of the triangle **abc**, is equal (i. 32) to the two angles **abc**, **acb**; therefore the angle **bac** is equal to the angle **fac**, and each of them is therefore a right angle (i. *def.* 10). Wherefore the angle **bac** in a semicircle is a right angle.



And because the two angles **abc**, **bac** of the triangle **abc** are together less (i. 17) than two right angles, and that **bac** is a right angle, **abc** must be less than a right angle; and therefore the angle in a segment **abc** greater than a semicircle, is less than a right angle.

And because **abcd** is a quadrilateral figure in a circle, any two of its opposite angles are equal (iii. 22) to two right angles: therefore the angles **abc**, **adc** are equal to two right angles; and **abc** is less than a right angle; wherefore the other **adc** is greater than a right angle.

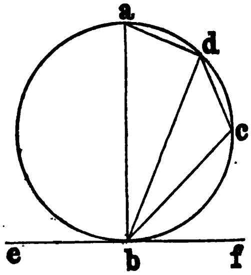
Besides, it is manifest that the circumference of the greater segment **abc** falls without the right angle **cab**; but the circumference of the less segment **adc** falls within the right angle **caf**. And this is all that is meant, when in the Greek text, and the translations from it, the angle of the greater segment is said to be greater, and the angle of the less segment is said to be less, than a right angle.

COR. From this it is manifest that if one angle of a triangle be equal to the other two it is a right angle, because the angle adjacent to it is equal to the same two; and when the adjacent angles are equal they are right angles.

PROPOSITION XXXII.—THEOREM.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

LET the straight line **e f** touch the circle **a b c d** in **b**, and from the point **b** let the straight line **b d** be drawn, cutting the circle. The angles which **b d** makes with the touching line **e f** shall be equal to the angles in the alternate segments of the circle: that is, the angle **f b d** is equal to the angle which is in the segment **d a b**, and the angle **d b e** to the angle in the segment **b c d**.

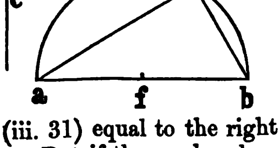


From the point **b** draw (i. 11) **b a** at right angles to **e f**, and take any point **c** in the circumference **b d**, and join **a d**, **d c**, **c b**; and because the straight line **e f** touches the circle **a b c d** in the point **b**, and **b a** is drawn at right angles to the touching line from the point of contact **b**, the centre of the circle is in **b a** (iii. 19); therefore the angle **a d b** in a semicircle is a right (iii. 31) angle, and consequently the other two angles **b a d**, **a b d** are equal (i. 32) to a right angle. But **a b f** is likewise a right angle: therefore the angle **a b f** is equal to the angles **b a d**, **a b d**. Take from these equals the common angle **a b d**; therefore the remaining angle **d b f** is equal to the angle **b a d**, which is in the alternate segment of the circle; and because **a b c d** is a quadrilateral figure in a circle, the opposite angles **b a d**, **b c d** are equal to two right angles (iii. 23): therefore the angles **d b f**, **d b e**, being likewise equal (i. 13) to two right angles, are equal to the angles **b a d**, **b c d**: and **d b f** has been proved equal to **b a d**. Therefore the remaining angle **d b e** is equal to the angle **b c d** in the alternate segment of the circle. Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION XXXIII.—PROBLEM.

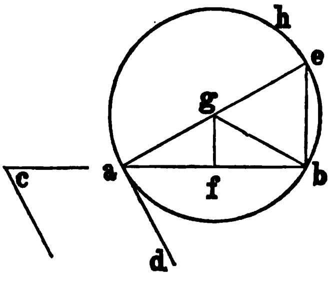
Upon a given straight line to describe a segment of a circle containing an angle equal to a given rectilineal angle.

LET **a b** be the given straight line, and the angle at **c** the given rectilineal angle; it is required to describe upon the given straight line **a b** a segment of a circle, containing an angle equal to the angle **c**.

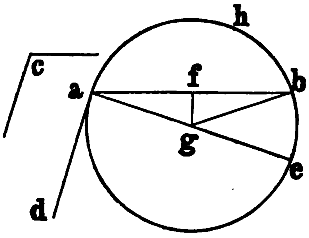


First, let the angle at **c** be a right angle, and bisect (i. 10) **a b** in **f**, and from the centre **f**, at the distance **f b**, describe the semicircle **a h b**; therefore the angle **a h b** in a semicircle is (iii. 31) equal to the right angle at **c**.

But if the angle **c** be not a right angle, at the point **a**, in the straight



line **a b**, make (i. 23) the angle **b a d** equal to the angle **c**, and from the point **a** draw (i. 11) **a e** at right angles to **a d**; bisect (i. 10) **a b** in **f**, and from **f** draw (i. 11) **f g** at right angles to **a b**, and join **g b**: and because **a f** is equal to **f b**, and **f g** common to the triangles **a f g**, **b f g**, the two sides **a f**, **f g** are equal to the two **b f**, **f g**; and the angle **a f g** is equal to the angle **b f g**; therefore the base **a g** is equal (i. 4) to the base **g b**: and the circle described from the centre **g**, at the distance **g a**, shall pass through the point **b**; let this be the circle **a h b**: and because from the point **a**, the extremity of the diameter **a e**, **a d** is drawn at right angles to **a e**, therefore **a d** (iii. 16. *cor.*) touches the circle; and because **a b** drawn from the point of contact **a** cuts the circle, the angle **d a b** is equal to the angle in the alternate segment **a h b** (iii. 32): but the angle **d a b** is equal to the angle **c**, therefore also the angle **c** is equal to the angle in the segment **a h b**: wherefore upon the given straight line **a b** the segment **a h b** of a circle is described, which contains an angle equal to the given angle at **c**. Which was to be done.

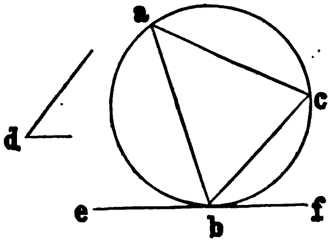


Proposition XXXIV.—Problem.

To cut off a segment from a given circle, which shall contain an angle equal to a given rectilineal angle.

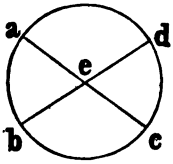
Let **a b c** be the given circle, and **d** the given rectilineal angle; it is required to cut off a segment from the circle **a b c** that shall contain an angle equal to the given angle **d**.

Draw (iii. 17) the straight line **e f** touching the circle **a b c** in the point **b**, and at the point **b**, in the straight line **b f**, make (i. 23) the angle **f b c** equal to the angle **d**: therefore, because the straight line **e f** touches the circle **a b c**, and **b c** is drawn from the point of contact **b**, the angle **f b c** is equal (iii. 32) to the angle in the alternate segment **b a c** of the circle: but the angle **f b c** is equal to the angle **d**: therefore the angle in the segment **b a c** is equal to the angle **d**: wherefore the segment **b a c** is cut off from the given circle **a b c**, containing an angle equal to the given angle **d**. Which was to be done.



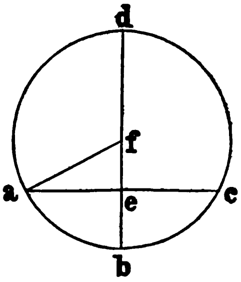
PROPOSITION XXXV.—THEOREM.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

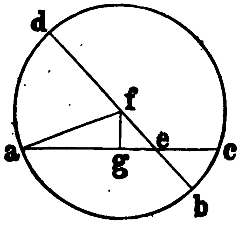


LET the two straight lines **ac**, **bd**, within the circle **abcd**, cut one another in the point **e**; the rectangle contained by **ae**, **ec** is equal to the rectangle contained by **be**, **ed**.

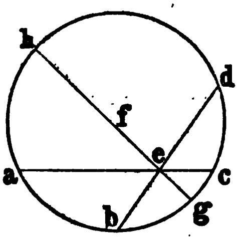
If **ac**, **bd** pass each of them through the centre, so that **e** is the centre, it is evident that **ae**, **ec**, **be**, **ed** being all equal, the rectangle **ae**, **ec**, is likewise equal to the rectangle **be**, **ed**.



But let one of them **bd** pass through the centre, and cut the other **ac** which does not pass through the centre, at right angles, in the point **e**: then, if **bd** be bisected in **f**, **f** is the centre of the circle **abcd**; join **af**. And because **bd**, which passes through the centre, cuts the straight line **ac**, which does not pass through the centre, at right angles in **e**; **ae**, **ec** are equal (iii. 3) to one another: and because the straight line **bd** is cut into two equal parts in the point **f**, and into two unequal in the point **e**, the rectangle **be**, **ed**, together, with the square of **ef**, is equal (ii. 5) to the square of **fb**; that is, to the square of **fa**: but the squares of **ae**, **ef**, are equal (i. 47) to the square of **fa**: therefore the rectangle **be**, **ed**, together with the square of **ef**, is equal to the squares of **ae**, **ef**: take away the common square of **ef**, and the remaining rectangle **be**, **ed** is equal to the remaining square of **ae**; that is, to the rectangle **ae**, **ec**.



Next, let **bd**, which passes through the centre, cut the other **ac**, which does not pass through the centre, in **e**, but not at right angles: then, as before, if **bd** be bisected in **f**, **f** is the centre of the circle. Join **af**, and from **f** draw (i. 12) **fg** perpendicular to **ac**; therefore **ag** is equal (iii. 3) to **gc**; wherefore the rectangle **ae**, **ec**, together with the square of **eg**, is equal (ii. 5) to the square of **ag**: to each of these equals add the square of **gf**; therefore the rectangle **ae**, **ec**, together with the squares of **eg**, **gf**, is equal to the squares of **ag**, **gf**: but the squares of **eg**, **gf**, are equal (i. 47) to the square of **ef**; and the squares of **ag**, **gf**, are equal to the square of **af**: therefore the rectangle **ae**, **ec**, together with the square of **ef**, is equal to the square of **af**; that is, to the square of **fb**: but the square of **fb** is equal (ii. 5) to the rectangle **be**, **ed**, together with the square of **ef**; therefore the rectangle **ae**, **ec**, together with the square of **ef**, is equal to the rectangle **be**, **ed**, together with the square of **ef**: take away the common square of **ef**, and the remaining rectangle **ae**, **ec**, is therefore equal to the remaining rectangle **be**, **ed**.

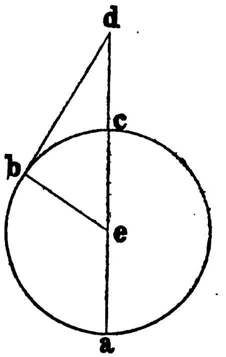


Lastly, let neither of the straight lines **a c**, **b d** pass through the centre: take the centre **f**, and through **e**, the intersection of the straight lines **a c**, **d b**, draw the diameter **g e f h**: and because the rectangle **a e**, **e c**, is equal, as has been shewn, to the rectangle **g e**, **e h**; and, for the same reason, the rectangle **b e**, **e d** is equal to the same rectangle **g e**, **e h**; therefore the rectangle **a e**, **e c** is equal to the rectangle **b e**, **e d**. Wherefore, if two straight lines, &c. Q. E. D.

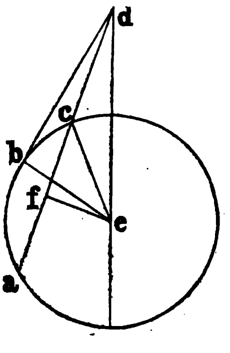
PROPOSITION XXXVI.—THEOREM.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it, the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

LET **d** be any point without the circle **a b c**, and **d c a**, **d b** two straight lines drawn from it, of which **d c a** cuts the circle, and **d b** touches the same: the rectangle **a d**, **d c** is equal to the square of **d b**.

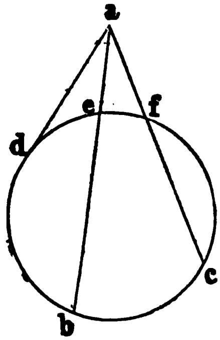


Either **d c a** passes through the centre, or it does not; first, let it pass through the centre **e**, and join **e b**; therefore the angle **e b d** is a right (iii. 18) angle; and because the straight line **a c** is bisected in **e**, and produced to the point **d**, the rectangle **a d**, **d c**, together with the square of **e c**, is equal (ii. 6) to the square of **e d**, and **c e** is equal to **e b**: therefore the rectangle **a d**, **d c**, together with the square of **e b**, is equal to the square of **e d**. But the square of **e d** is equal (i. 47) to the squares of **e b**, **b d**, because **e b d** is a right angle: therefore the rectangle **a d**, **d c**, together with the square of **e b**, is equal to the squares of **e b**, **b d**. Take away the common square of **e b**; therefore the remaining rectangle **a d**, **d c**, is equal to the square of the tangent **d b**.



But if **d c a** does not pass through the centre of the circle **a b c**, take (iii. 1) the centre **e**, and draw **e f** perpendicular (i. 12) to **a c**, and join **e b**, **e c**, **e d**: and because the straight line **e f**, which passes through the centre, cuts the straight line **a c**, which does not pass through the centre, at right angles, it shall likewise bisect (iii. 3) it; therefore **a f** is equal to **f c**: and because the straight line **a c** is bisected in **f**, and produced to **d**, the rectangle **a d**, **d c**, together with the square of **f c**, is equal (ii. 6) to the square of **f d**. To each of these equals add the square of **f e**; theretore the rectangle **a d**, **d c**, together with the squares of **c f**, **f e**, is equal to the squares of **d f**, **f e**; but the square of **e d** is equal

(i. 47) to the squares of **d f**, **f e**, because **e f d** is a right angle: and the square of **e c** is equal to the squares of **c f**, **f e**; therefore the rectangle **a d**, **d c**, together with the square of **e c**, is equal to the square of **e d**: and **c e** is equal to **e b**; therefore the rectangle **a d**, **d c**, together with the square of **e b**, is equal to the square of **e d**; but the squares of **e b**, **b d** are equal to the square (i. 47) of **e d**, because **e b d** is a right angle; therefore the rectangle **a d**, **d c**, together with the square of **e b**, is equal to the squares of **e b**, **b d**. Take away the common square of **e b**; therefore the remaining rectangle **a d**, **d c** is equal to the square of **d b**. Wherefore, if from any point, &c. Q. E. D.



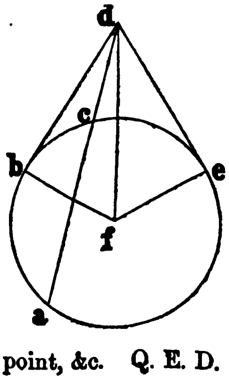
COR. If from any point without a circle, there be drawn two straight lines cutting it, as **a b**, **a c**, the rectangles contained by the whole lines, and the parts of them without the circle, are equal to one another, viz. the rectangle **b a**, **a e**, to the rectangle **c a**, **a f**: for each of them is equal to the square of the straight line **a d**, which touches the circle.

PROPOSITION XXXVII.—THEOREM.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square of the line which meets it, the line which meets shall touch the circle.

LET any point **d** be taken without the circle **a b c**, and from it let two straight lines **d c a** and **d b** be drawn, of which **d c a** cuts the circle, and **d b** meets it; if the rectangle **a d**, **d c** be equal to the square of **d b**; **d b** touches the circle.

Draw (iii. 17) the straight line **d e** touching the circle **a b c**, find its centre **f**, and join **f e**, **f b**, **f d**; then **f e d** is a right (iii. 18) angle: and because **d e** touches the circle **a b c**, and **d c a** cuts it, the rectangle **a d**, **d c** is equal (iii. 36) to the square of **d e**: but the rectangle **a d**, **d c** is, by hypothesis, equal to the square of **d b**: therefore the square of **d e** is equal to the square of **d b**; and the straight line **d e** equal to the straight line **d b**; and **f e** is equal to **f b**, wherefore **d e**, **e f** are equal to **d b**, **b f**; and the base **f d** is common to the two triangles **d e f**, **d b f**; therefore the angle **d e f** is equal (i. 8) to the angle **d b f**; but **d e f** is a right angle, therefore also **d b f** is a right angle: and **f b**, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches (iii. 16) the circle: therefore **b d** touches the circle **a b c**. Wherefore, if from a point, &c. Q. E. D.



EXERCISES ON BOOK III.

SECT. I.—PROBLEMS.

1. Given a circle, a square, and a point without the circle, to draw from the point a straight line cutting the circle, such that the rectangle contained by the part of the line within, and the part without the circle, shall be equal to the given square.

2. Given a circle and a point within it which is not the centre, to draw through that point a chord which will be bisected in the point.

3. Given a circle with its diameter produced to a given point, to find in the part produced a point, such that if a tangent be drawn from it to the circle, the tangent shall be equal to the part of the produced diameter lying between the point and the given point.

4. Given a circle with the diameter produced and a straight line, to draw a tangent from the diameter produced to the circle, equal to the given straight line.

5. Given two circles, to draw a straight line which shall touch them.

6. Given the vertical angle, the altitude and the sum of the three sides of a triangle, to construct the triangle.

7. Given the vertical angle, the altitude and the base of a triangle, to construct the triangle.

8. Given a segment of a circle and a straight line, to describe on the line another segment of a circle which shall be similar to the given segment.

9. Given a point, a straight line, and a point in the line, to describe a circle which shall pass through the first point, and touch the straight line in the other point.

SECT. II.—THEOREMS.

10. The line drawn from the centre of a circle perpendicular to any chord is also perpendicular to all chords parallel to the former.

11. If two parallel chords in a circle are bisected, the bisecting line is also perpendicular to them.

12. If from any two points in the circumference of the greater of two

concentric circles, chords are drawn so as to touch the circumference of the lesser circle, these chords shall be equal.

13. If two parallel chords are drawn in a circle, the arcs intercepted between their extremities shall be equal.

14. If any point be taken without a circle, there cannot be drawn from that point more than two equal straight lines to touch the circumference of the circle.

15. If an equilateral triangle be inscribed in a circle, and from a point in the circumference straight lines be drawn to the three angles of the triangle, the greatest of these lines is equal to the sum of the other two.

16. If a quadrilateral be described about a circle, the angles subtended by any two opposite sides of the figure at the centre of the circle shall be equal to two right angles.

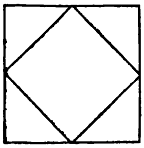
17. If two circles cut each other, the straight line which joins their centres will bisect the straight line which joins the points of intersection.

BOOK IV.

DEFINITIONS.

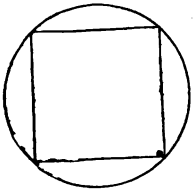
I. A **RECTILINEAL** figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

II. In like manner a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



Definitions I. and II.

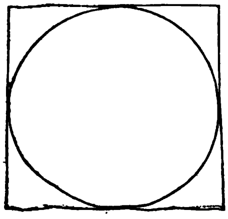
III. A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



Definition III.

IV. A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

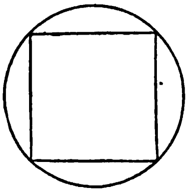
V. In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.



Definitions IV. and V.

VI. A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

VII. A straight line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

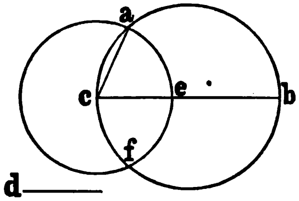


Definitions VI. and VII.

PROPOSITION I.—PROBLEM.

In a given circle to place a straight line equal to a given straight line not greater than the diameter of the circle.

LET **a b c** be the given circle, and **d** the given straight line, not greater than the diameter of the circle.

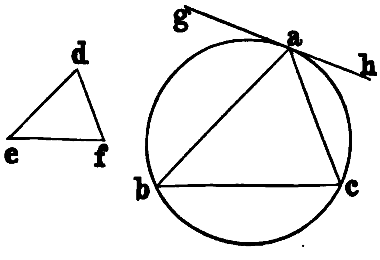


Draw **b c** the diameter of the circle **a b c**; then, if **b c** is equal to **d**, the thing required is done, for in the circle **a b c** a straight line **b c** is placed equal to **d**; but, if it is not, **b c** is greater than **d**; make **c e** equal (i. 3) to **d**, and from the centre **c**, at the distance **c e**, describe the circle **a e f**, and join **c a**: therefore, because **c** is the centre of the circle **a e f**, **c a** is equal to **c e**: but **d** is equal to **c e**; therefore **d** is equal to **c a**. Wherefore in the circle **a b c**, a straight line is placed equal to the given straight line **d**, which is not greater than the diameter of the circle. Which was to be done.

PROPOSITION II.—PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.

LET **a b c** be the given circle, and **d e f** the given triangle; it is required to inscribe, in the circle **a b c** a triangle equiangular to the triangle **d e f**.



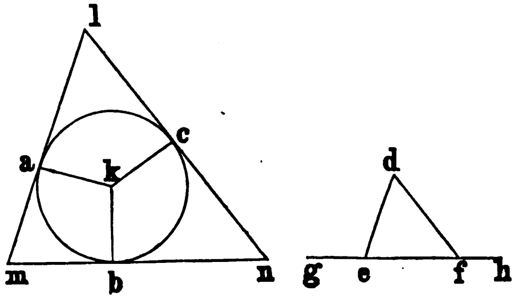
Draw (iii. 17) the straight line **g a h** touching the circle in the point **a**, and at the point **a**, in the straight line **a h**, make (i. 23) the angle **h a c** equal to the angle **d e f**; and at the point **a**, in the straight line **a g**, make the angle **g a b** equal to the angle **dfe**, and join **bc**: therefore because **hag** touches the circle **abc**, and **ac** is drawn from the point of contact, the angle **hac** is equal (iii. 32) to the angle **abc** in the alternate segment of the circle; but **hac** is equal to the angle **def**; therefore also the angle **abc** is equal to **def**. For the same reason, the angle **acb** is equal to the angle **dfe**; therefore the remaining angle **bac** is equal (i. 32) to the remaining angle **edf**: wherefore the triangle **abc** is equiangular to the triangle **def**, and it is inscribed in the circle **abc**. Which was to be done.

PROPOSITION III.—PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

LET **abc** be the given circle, and **def** the given triangle; it is required to describe a triangle about the circle **abc** equiangular to the triangle **def**.

Produce **ef** both ways to the points **g**, **h**, and find the centre **k** of the circle **abc**, and from it draw any straight line **kb**; at the point **k** in the straight line **kb**, make (i. 23) the angle **bka** equal to the angle **deg**, and the angle **bkc** equal to the angle **dfh**; and through the points **a**, **b**, **c**, draw the straight lines **lam**, **mbn**, **ncl**, touching (iii. 17) the circle **abc**: therefore, because **lm**, **mn**, **nl** touch the circle **abc** in the points **a**, **b**, **c**, to which from the centre are drawn **ka**, **kb**, **kc**, the angles at the points **a**, **b**, **c**, are right (iii. 18) angles: and because the four angles of the quadrilateral figure **ambk** are equal to four right angles, for it can be divided into two triangles; and that two of them **kam**, **kbm** are right angles, the other two **akb**, **amb** are equal to two right angles: but the angles **deg**, **def** are likewise equal (i. 13) to two right angles; therefore the angles **akb**, **amb** are equal to the angles **deg**, **def**, of which **akb** is equal to **deg**; wherefore the remaining angle **amb** is equal to the remaining angle **def**. In like manner, the angle **lnm** may be demonstrated to be equal to **dfe**; and therefore the remaining angle **mln** is equal (i. 32) to the remaining angle **edf**: wherefore the triangle **lmn** is equiangular to the triangle **def**: and it is described about the circle **abc**. Which was to be done.

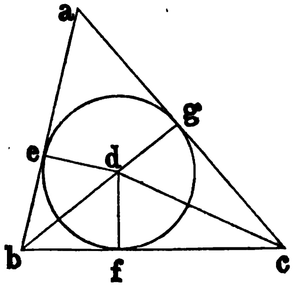


PROPOSITION IV.—PROBLEM.

To inscribe a circle in a given triangle.

LET the given triangle be **a b c**; it is required to inscribe a circle in **a b c**.

Bisect (i. 9) the angles **a b c**, **b c a** by the straight lines **b d**, **c d** meeting one another in the point **d**, from which draw (i. 12) **d e**, **d f**, **d g** perpendiculars to **a b**, **b c**, **c a**: and because the angle **e b d** is equal to the angle **f b d**, for the angle **a b c** is bisected by **b d**, and that the right angle **b e d** is equal to the right angle **b f d**, the two triangles **e b d**, **f b d** have two angles of the one equal to two angles of the other, and the side **b d**, which is opposite to one of the equal angles in each, is common to both; therefore their other sides shall be equal (i. 26); wherefore **d e** is equal to **d f**. For the same reason, **d g** is equal to **d f**; therefore the three straight lines **d e**, **d f**, **d g**, are equal to one another, and the circle described from the centre **d**, at the distance of any of them, shall pass through the extremities of the other two, and touch the straight lines **a b**, **b c**, **c a**. Because the angles at the points **e**, **f**, **g** are right angles, and the straight line which is drawn from the extremity of a diameter at right angles to it, touches (iii. 13) the circle: therefore the straight lines **a b**, **b c**, **c a** do each of them touch the circle, and the circle **e f g** is inscribed in the triangle **a b c**. Which was to be done.

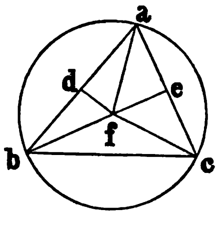


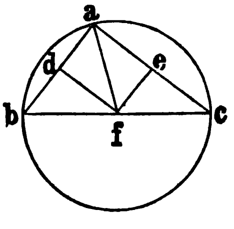
PROPOSITION V.—PROBLEM.

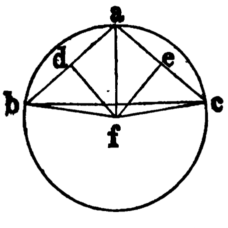
To describe a circle about a given triangle.

LET the given triangle be **a b c**; it is required to describe a circle about **a b c**.

Bisect (i. 10) **a b**, **a c** in the points **d**, **e**, and from these points draw **d f**, **e f** at right angles (i. 11) to **a b**, **a c**; **d f**, **e f**, produced, meet one another. For, if they do not meet, they are parallel, wherefore **a b**, **a c**, which are at right angles to them, are parallel; which is absurd. Let them meet in **f**, and join **f a**; also if the point **f** be not in **b c**, join **b f**, **c f**: then, because **a d** is equal to **d b**, and **d f** common, and at right angles







to **a b**, the base **a f** is equal (i. 4) to the base **f b**. In like manner, it may be shewn that **c f** is equal to **f a**; and therefore **b f** is equal to **f c**; and **f a**, **f b**, **f c**, are equal to one another; wherefore the circle described from the centre **f**, at the distance of one of them, shall pass through the extremities of the other two, and be described about the triangle **a b c**. Which was to be done.

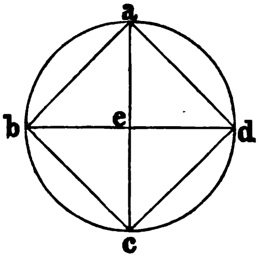
COR. And it is manifest, that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but, when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and, if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle: wherefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the side opposite to the right angle; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROPOSITION VI.—PROBLEM.

To inscribe a square in a given circle.

LET **a b c d** be the given circle; it is required to inscribe a square in **a b c d**.

Draw the diameters **a c**, **b d**, at right angles to one another, and join **a b**, **b c**, **c d**, **d a**; because **b e** is equal to **e d**, for **e** is the centre, and that **e a** is common, and at right angles to **b d**; the base **b a** is equal (i. 4) to the base **a d**; and, for the same reason, **b c**, **c d** are each of them equal to **b a**, or **a d**; therefore the quadrilateral figure **a b c d** is equilateral. It is also rectangular; for the straight line **b d**, being the diameter of the circle **a b c d**, **b a d** is a semicircle; wherefore the angle **b a d** is a right (iii. 31) angle; for the same reason, each of the angles **a b c**, **b c d**, **c d a**, is a right angle; therefore the quadrilateral figure **a b c d** is rectangular, and it has been shewn to be equilateral; therefore it is a square; and it is inscribed in the circle **a b c d**. Which was to be done.

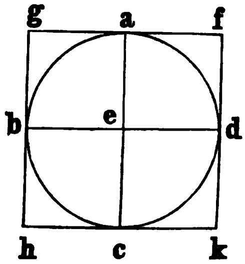


PROPOSITION VII.—PROBLEM.

To describe a square about a given circle.

LET **a b c d** be the given circle; it is required to describe a square about it.

Draw two diameters **a c**, **b d** of the circle **a b c d**, at right angles to one another, and through the points **a**, **b**, **c**, **d**, draw (iii. 17) **f g**, **g h**, **h k**, **k f**, touching the circle; and because **f g** touches the circle **a b c d**, and **e a** is drawn from the centre **e** to the point of contact **a**, the angles at



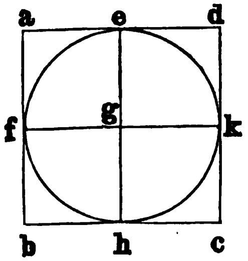
a are right angles (iii. 18); for the same reason, the angles at the points **b, c, d,** are right angles; and because the angle **a e b** is a right angle, as likewise is **e b g**, **g h** is parallel (i. 28) to **a c**; for the same reason **a c** is parallel to **f k**, and in like manner **g f**, **h k** may each of them be demonstrated to be parallel to **b e d**; therefore the figures **g k**, **g c**, **a k**, **f b**, **b k** are parallelograms; and **g f** is therefore equal (i. 34) to **h k**, and **g h** to **f k**; and because **a c** is equal to **b d**, and that **a c** is equal to each of the two **g h**, **f k**; and **b d** to each of the two **g f**, **h k**: **g h**, **f k** are each of them equal to **g f** or **h k**; therefore the quadrilateral figure **f g h k** is equilateral. It is also rectangular; for **g b e a** being a parallelogram, and **a e b** a right angle, **a g b** (i. 34) is likewise a right angle. In the same manner it may be shewn that the angles at **h**, **k**, **f**, are right angles; therefore the quadrilateral figure **f g h k** is rectangular, and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle **a b c d**. Which was to be done.

Proposition VIII.—Problem.

To inscribe a circle in a given square.

Let **a b c d** be the given square; it is required to inscribe a circle in **a b c d**.

Bisect (i. 10) each of the sides **a b**, **a d** in the points **f**, **e**, and through **e** draw (i. 31) **e h** parallel to **a b** or **d c**, and through **f** draw **f k** parallel to **a d** or **b c**; therefore each of the figures **a k**, **k b**, **a h**, **h d**, **a g**, **g c**, **b g** **g d**, is a parallelogram, and their opposite sides are equal (i. 34): and because **a d** is equal to **a b**, and that **a e** is the half of **a d**, and **a f** the half of **a b**, **a e** is equal to **a f**; wherefore the sides opposite to these are equal, viz. **f g** to **g e**; in the same manner it may be demonstrated that **g h**, **g k** are each of them equal to **f g** or **g e**: therefore the four straight lines **g e**, **g f**, **g h**, **g k** are equal to one another; and the circle described from the centre **g** at the distance of one of them, shall pass through the extremities of the other three, and touch the straight lines **a b**, **b c**, **c d**, **d a**: because the angles at the points **e**, **f**, **h**, **k**, are right (i. 29) angles, and that the straight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle (iii. 16); therefore each of the straight lines **a b**, **b c**, **c d**, **d a**, touches the circle, which therefore is inscribed in the square **a b c d**. Which was to be done.

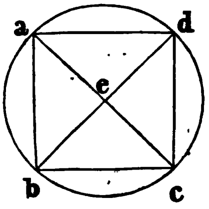


PROPOSITION IX.—PROBLEM.

To describe a circle about a given square.

LET a b c d be the given square; it is required to describe a circle about it.

Join a c, b d, cutting one another in e; and because d a is equal to a b, and a c common to the triangles d a c, b a c, the two sides d a, a c, are equal to the two b a, a c, and the base d c is equal to the base b c; wherefore the angle d a c is equal (i. 8) to the angle b a c, and the angle d a b is bisected by the straight line a c. In the same manner it may be demonstrated that the angles a b c, b c d, c d a are severally bisected by the straight lines b d, a c; therefore, because the angle d a b is equal to the angle a b c, and that the angle e a b is the half of d a b, and e b a the half of a b c: the angle e a b is equal to the angle e b a; wherefore the side e a is equal (i. 6) to the side e b. In the same manner it may be demonstrated that the straight lines e c, e d are each of them equal to e a or e b; therefore the four straight lines e a, e b, e c, e d, are equal to one another, [illegible]d the circle described from the centre e, at the distance of one of [illegible] shall pass through the extremities of the other three, and be des[illegible]d about the square a b c d. Which was to be done.

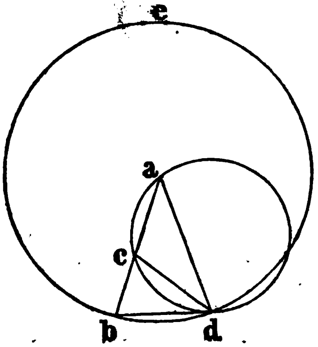


PROPOSITION X.—PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

TAKE any straight line a b, and divide (ii. 11) it in the point c, so that the rectangle a b, b c be equal to the square of c a; and from the centre a, at the distance a b, describe the circle b d e, in which place (iv. 1) the straight line b d equal to a c, which is not greater than the diameter of the circle b d e; join d a, d c, and about the triangle a d c describe (iv. 5) the circle a c d; the triangle a b d is such as is required, that is, each of the angles a b d, a d b is double of the angle b a d.

Because the rectangle a b, b c is equal to the square of a c, and that a c is equal to b d, the rectangle a b, b c is equal to the square of b d; and because from the point b, without the circle a c d, two straight lines b c a, b d are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle a b, b c, contained by the whole of the cutting line, and the part of it without the circle, is equal to the square of b d, which meets it; the straight line b d touches (iii. 37) the circle a c d; and be-



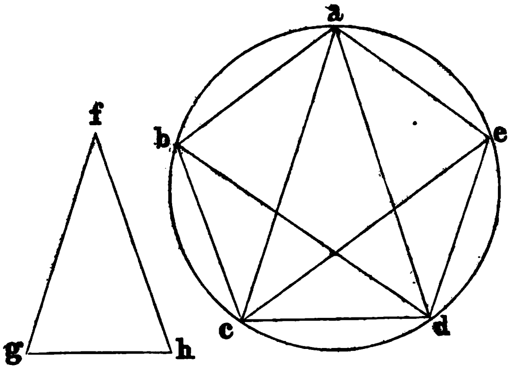
cause **bd** touches the circle, and **dc** is drawn from the point of contact **d**, the angle **bdc** is equal (iii. 32) to the angle **dac** in the alternate segment of the circle; to each of these add the angle **cda**; therefore the whole angle **bda** is equal to the two angles **cda**, **dac**; but the exterior angle **bcd** is equal (i. 32) to the angles **cda**, **dac**; therefore also **bda** is equal to **bcd**; but **bda** is equal (i. 5) to the angle **cbd**, because the side **ad** is equal to the side **ab**; therefore **cbd**, or **dba** is equal to **bcd**; and consequently, the three angles **bda**, **dba**, **bcd**, are equal to one another; and because the angle **dbc** is equal to the angle **bcd**, the side **bd** is equal (i. 6) to the side **dc**; but **bd** was made equal to **ca**, therefore also **ca** is equal to **cd**, and the angle **cda** equal (i. 5) to the angle **dac**; therefore the angles **cda**, **dac** together, are double of the angle **dac**: but **bcd** is equal to the angles **cda**, **dac**, therefore also **bcd** is double of **dac**, and **bcd** is equal to each of the angles **bda**, **dba**; each therefore of the angles **bda**, **dba** is double of the angle **dab**, wherefore an isosceles triangle **abd** is described, having each of the angles at the base double of the third angle. Which was to be done.

PROPOSITION XI.—PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

LET **abcde** be the given circle; it is required to inscribe an equilateral and equiangular pentagon in the circle **abcde**.

Describe (iv. 10) an isosceles triangle **fgh**, having each of the angles at **g**, **h** double of the angle at **f**; and in the circle **abcde** inscribe (iv. 2) the triangle **acd** equiangular to the triangle **fgh**, so that the angle **cad** be equal to the angle at **f**, and each of the angles **acd**, **cda** equal to the angle at **g** or **h**, wherefore each of the angles **acd**, **cda** is double of the angle **cad**. Bisect (i. 9) the angles **acd**, **cda** by the straight lines **ce**, **db**; and join **ab**, **bc**, **de**, **ea**. **abcde** is the pentagon required.



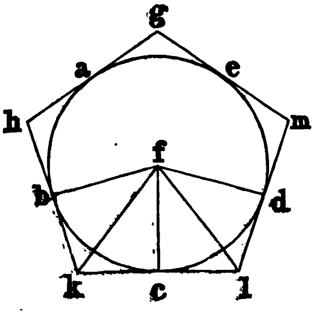
Because each of the angles **acd**, **cda** is double of **cad**, and are bisected by the straight lines **ce**, **db**, the five angles **dac**, **ace**, **ecd**, **cdb**, **bda** are equal to one another; but equal angles stand upon equal circumferences (iii. 26); therefore the five circumferences, **ab**, **bc**, **cd**, **de**, **ea** are equal to one another: and equal circumferences are subtended by equal straight lines (iii. 29); therefore the five straight lines **ab**, **bc**, **cd**, **de**, **ae** are equal to one another. Wherefore the pentagon **abcde** is equilateral. It is also equiangular; because the circumference **ab** is equal to the circumference **de**; if to each be added **bcd**, the whole **abcd** is equal to the whole **edcb**: and the angle **aed** stands on the circumference **abcd**, and the angle **bae** on the circumference **edcb**; therefore the angle **bae** is equal (iii. 27) to the angle **aed**. For the same reason, each of the angles **abc**, **bcd**, **cde** is equal to the angle **bae**, or **aed**; therefore the pentagon **abcde** is equiangular; and it has been shewn that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed. Which was to be done.

PROPOSITION XII.—PROBLEM.

To describe an equilateral and equiangular pentagon about a given circle.

LET **abcde** be the given circle; it is required to describe an equilateral and equiangular pentagon about the circle **abcde**.

Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points **a**, **b**, **c**, **d**, **e**, so that the circumferences **ab**, **bc**, **cd**, **de**, **ea** are equal (iv. 11); and through the points **a**, **b**, **c**, **d**, **e**, draw **gh**, **hk**, **kl**, **lm**, **mg**, touching (iii. 17) the circle; take the centre **f**, and join **fb**, **fk**, **fc**, **fl**, **fd**; and because the straight line **kl** touches the circle **abcde** in the point **c**, to which **fc** is drawn from the centre **f**, **fc** is perpendicular (iii. 18) to **kl**, therefore each of the angles at **c** is a right angle: for the same reason, the angles at the points **b**, **d** are right angles: and because **fck** is a right angle, the square of **fk** is equal (i. 47) to the squares of **fc**, **ck**: for the same reason, the square of **fk** is equal to the squares of **fb**, **bk**: therefore the squares of **fc**, **ck** are equal to the squares of **fb**, **bk**, of which the square of **fc** is equal to the square of **fb**; the remaining square of **ck** is therefore equal to the remaining square of **bk**, and the straight line **ck** equal to **bk**: and because **fb** is equal to **fc**, and **fk** common to the triangles **bfk**, **cfk**, the two **bf**, **fk** are equal to the two **cf**, **fk**, and the base **bk** is equal to the base **kc**; therefore the angle **bfk** is equal (i. 8) to the angle **kfc**, and the angle **bkf** to **fkc**; wherefore the angle **bfc** is double of the angle **kfc**, and **bkc** double of **fkc**: for the same reason, the angle **cfd** is double of the angle **cfl**, and **cld** double of **clf**: and because the circumference **bc** is equal to the circumference **cd**, the angle **bfc** is equal (iii. 27) to the angle **cfd**; and **bfc** is double of the angle **kfc**, and **cfd** double of **cfl**; therefore the angle **kfc** is equal to the angle **cfl**; and the right angle **fck** is equal to the right angle **fcl**: therefore, in the two triangles **fkc**, **flc**, there are two angles of one equal to two angles of the other, each to each, and the side **fc**, which is adjacent to the equal angles in each, is common to both; therefore the other sides shall be equal (i. 26) to the other sides, and the third angle to the third angle:

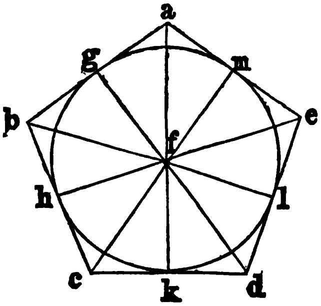


therefore the straight line **kc** is equal to **cl**, and the angle **fkc** to the angle **flc**: and because **kc** is equal to **cl**, **kl** is double of **kc**. In the same manner it may be shewn that **hk** is double of **bk**: and because **bk** is equal to **kc**, as was demonstrated, and that **kl** is double of **kc**, and **kh** double of **bk**, **hk** shall be equal to **kl**. In like manner, it may be shewn that **gh**, **gm**, **ml** are each of them equal to **hk**, or **kl**: therefore the pentagon **ghklm** is equilateral. It is also equiangular; for, since the angle **fkc** is equal to the angle **flc**, and that the angle **hkl** is double of the angle **fkc**, and **klm** double of **flc**, as was before demonstrated, the angle **hkl** is equal to **klm**: and in like manner it may be shewn that each of the angles **khg**, **hgm**, **gml** is equal to the angles **hkl** or **klm**: therefore the five angles **ghk**, **hkl**, **klm**, **lmg**, **mgh**, being equal to one another, the pentagon **ghklm** is equiangular; and it is equilateral, as was demonstrated; and it is described about the circle **abcde**. Which was to be done.

PROPOSITION XIII.—PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon.

LET **abcde** be the given equilateral and equiangular pentagon; it is required to inscribe a circle in the pentagon **abcde**.



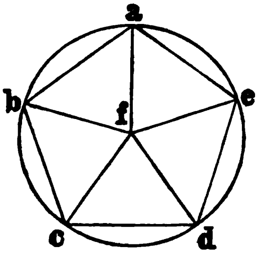
Bisect (i. 9) the angles **bcd**, **cde** by the straight lines **cf**, **df**, and from the point **f**, in which they meet, draw the straight lines **fb**, **fa**, **fe**: therefore, since **bc** is equal to **cd**, and **cf** common to the triangles **bcf**, **dcf** the two sides **bc**, **cf** are equal to the two **dc**, **cf**; and the angle **bcf** is equal to the angle **dcf**; therefore the base **bf** is equal (i. 4) to the base **fd**, and the other angles to the other angles, to which the equal sides are opposite; therefore the angle **cbf** is equal to the angle **cdf**: and because the angle **cde** is double of **cdf**, and that **cde** is equal to **cba**, and **cdf** to **cbf**; **cba** is also double of the angle **cbf**; therefore the angle **abf** is equal to the angle **cbf**; wherefore the angle **abc** is bisected by the straight line **bf**. In the same manner, it may be demonstrated that the angles **bae**, **aed**, are bisected by the straight lines **af**, **fe**: from the point **f**, draw (i. 12) **fg**, **fh**, **fk**, **fl**, **fm** perpendiculars to the straight lines **ab**, **bc**, **cd**, **de**, **ea**: and because the angle **hcf** is equal to **kcf**, and the right angle **fhc** equal to the right angle **fkc**; in the triangles **fhc**, **fkc** there are two angles of one equal to two angles of the other, and the side **fc**, which is opposite to one of the equal angles in each, is common to both; therefore the other sides shall be equal (i. 26), each to each; wherefore the perpendicular **fh** is equal to the perpendicular **fk**. In the same manner, it may be demonstrated that **fl**, **fm**, **fg** are each of them equal to **fh**, or **fk**: therefore the five straight lines **fg**, **fh**, **fk**, **fl**, **fm** are equal to one another: wherefore the circle described from the centre **f**, at the distance of one of these five, shall pass through the extremities of the other four, and touch the straight lines **ab**, **bc**, **cd**, **de**, **ea**, because the angles at the points **g**, **h**, **k**, **l**, **m** are right angles; and that a straight line drawn from the extremity of the diameter of a circle at right angles to it, touches (iii. 16) the circle; therefore each of the straight lines **ab**, **bc**, **cd**, **de**, **ea** touches the circle: wherefore it is inscribed in the pentagon **abcde**. Which was to be done.

Proposition XIV.—Problem.

To describe a circle about a given equilateral and equiangular pentagon.

Let **abcde** be the given equilateral and equiangular pentagon; it is required to describe a circle about it.

Bisect (i. 9) the angles **bcd**, **cde** by the straight lines **cf**, **fd**, and from the point **f**, in which they meet, draw the straight lines **fb**, **fa**, **fe**, to the points **b**, **a**, **e**. It may be demonstrated in the same manner as in the preceding proposition, that the angles **cba**, **bae**, **aed** are bisected by the straight lines **fb**, **fa**, **fe**. And because the angle **bcd** is equal to the angle **cde**, and that **fcd** is the half of the angle **bcd**, and **cdf** the half of **cde**; the angle **fcd** is equal to **fdc**; wherefore the side **cf** is equal (i. 6) to the side **fd**. In like manner, it may be demonstrated that **fb**, **fa**, **fe** are each of them equal to **fc**, or **fd**. Therefore the five straight lines **fa**, **fb**, **fc**, **fd**, **fe** are equal to one another; and the circle described from the centre **f**, at the distance of one of them, shall pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon **abcde**. Which was to be done.



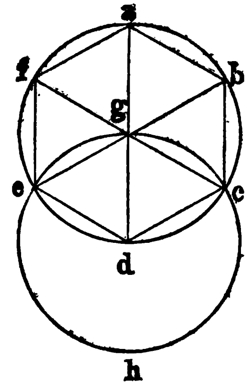
Proposition XV.—Problem.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let **abcdef** be the given circle; it is required to inscribe an equilateral and equiangular hexagon in it.

Find the centre **g** of the circle **abcdef**, and draw the diameter **agd**; and from **d** as a centre, at the distance **dg**, describe the circle **egch**, join **eg**, **cg**, and produce them to the points **b**, **f**; and join **ab**, **bc**, **cd**, **de**, **ef**, **fa**. The hexagon **abcdef** is equilateral and equiangular.

Because **g** is the centre of the circle **abcdef**, **ge** is equal to **gd**. And because **d** is the centre of the circle **egch**, **de** is equal to **dg**; wherefore **ge** is equal to **ed**, and the triangle **egd** is equilateral; and therefore its three angles **egd**, **gde**, **deg** are equal to one another, be-



cause the angles at the base of an isosceles triangle are equal (i. 5); and the three angles of a triangle are equal (i. 32) to two right angles; therefore the angle **egd** is the third part of two right angles. In the same manner it may be demonstrated that the angle **dgc** is also the third part of two right angles. And because the straight line **gc** makes with **eb** the adjacent angles **egc**, **cgb** equal (i. 13) to two right angles; the remaining angle **cgb** is the third part of two right angles: therefore the angles **egd**, **dgc**, **cgb** are equal to one another. And to these are equal (i. 15) the vertical opposite angles **bga**, **agf**, **fge**, therefore the six angles **egd**, **dgc**, **cgb**, **bga**, **agf**, **fge** are equal to one another. But equal angles stand upon equal (iii. 26) circumferences; therefore the six circumferences **ab**, **bc**, **cd**, **de**, **ef**, **fa** are equal to one another. And equal circumferences are subtended by equal (iii. 29) straight lines; therefore the six straight lines are equal to one another, and the hexagon **abcdef** is equilateral. It is also equiangular; for, since the circumference **af** is equal to **ed**, to each of these add the circumference **abcd**; therefore the whole circumference **fabcd** shall be equal to the whole **edcba**. And the angle **fed** stands upon the circumference **fabcd**, and the angle **afe** upon **edcba**; therefore the angle **afe** is equal to **fed**. In the same manner, it may be demonstrated that the other angles of the hexagon **abcdef** are each of them equal to the angle **afe** or **fed**: therefore the hexagon is equianglar; and it is equilateral, as was shewn; and it is inscribed in the given circle **abcdef**. Which was to be done.

Cor. From this it is manifest, that the side of the hexagon is equal to the straight line from the centre, that is, to the semidiameter of the circle.

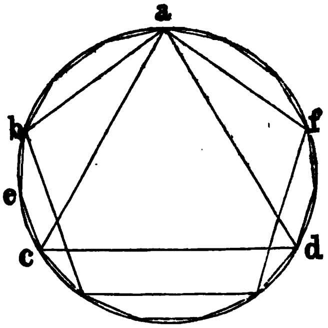
And if through the points **a**, **b**, **c**, **d**, **e**, **f** there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like to that used for the pentagon.

Proposition XVI.—Problem.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let **abcd** be the given circle; it is required to inscribe an equilateral and equiangular quindecagon in the circle **abcd**.

Let **ac** be the side of an equilateral triangle inscribed (ii. 4) in the circle, and **ab** the side of an equilateral and equiangular pentagon inscribed (iv. 11) in the same; therefore, of such equal parts as the whole circumference **abcdf** contains fifteen, the circumference **abc**, being the third part of the whole, contains five; and the circumference **a b**, which is the fifth part of the whole, contains three; therefore **b c** their difference contains two of the same parts. Bisect (iii. 30) **b c** in **e**; therefore **b e**, **e c** are, each of them, the fifteenth part of the whole circumference **a b c d**. Therefore, if the straight lines **b e**, **e c** be drawn, and straight lines equal to them be placed round (i. 4) in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it. Which was to be done.



And in the same manner as was done in the pentagon, if, through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it. And, likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

EXERCISES ON BOOK IV.

SECT. I.—PROBLEMS.

1. Given a line and two points, to describe a circle that shall pass through the points and touch the line.

2. Given a circle, to inscribe in it three equal circles which shall touch it and each other.

3. Given a circle, to inscribe in it four equal circles which shall touch it and each other.

4. Given a circle, to describe six other circles, each of which shall be equal to the given circle and be in contact with it and each other.

5. Given a quadrant of a circle, to inscribe a circle in it.

6. Given two circles, to draw a line that shall be a tangent to both.

7. Given a finite straight line, to describe on it an equilateral and equiangular decagon.

8. Given a line and a circle, to draw a tangent to the circle parallel to the line.

SECT. II.—THEOREMS.

9. If the three angles of a triangle be bisected, the bisecting lines shall meet in the same point.

10. If a circle be inscribed in a triangle, the rectangle contained by the radius of the circle and the sum of the three sides of the triangles is double of the triangle.

11. If an equilateral triangle be described about a circle, the straight lines which join the points of contact of the circle and the triangle contain another equilateral triangle, each of which is equal to one half of a side of the other triangle.

12. If a square be inscribed in a circle, it shall be double the square of the radius of the circle.

BOOK V.

DEFINITIONS.

I. A LESS magnitude is said to be a part of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

II. A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

III. Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity.

IV. Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V. The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI. Magnitudes which have the same ratio are called proportionals. N.B. When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second, as the third to the fourth.

VII. When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII. Analogy, or proportion, is the similitude of ratios.

IX. Proportion consists in three terms at least.

X. When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI. When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

Definition a, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if **a**, **b**, **c**, **d** be four magnitudes of the same kind, the first **a** is said to have to the last **d** the ratio compounded of the ratio of **a** to **b**, and the ratio of **b** to **c**, and the ratio of **c** to **d**; or, the ratio of **a** to **d** is said to be compounded of the ratios of **a** to **b**, **b** to **c**, and **c** to **d**:

And if **a** has to **b** the same ratio which **e** has to **f**; and **b** to **c** the same ratio that **g** has to **h**; and **c** to **d** the same that **k** has to **l**; then by this definition, **a** is said to have to **d** the ratio compounded of ratios which are the same with the ratios of **e** to **f**, **g** to **h**, and **k** to **l**. And the same thing is to be understood when it is more briefly expressed by saying, **a** has to **d** the ratio compounded of the ratios of **e** to **f**, **g** to **h**, and **k** to **l**.

In like manner, the same things being supposed, if **m** has to **n** the same ratio which **a** has to **d**; then, for shortness sake, **m** is said to have to **n** the ratio compounded of the ratios of **e** to **f**, **g** to **h**, and **k** to **l**.

XII. In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of proportionals, so as that they continue still to be proportionals.

XIII. *Permutando*, or *alternando*, by permutation, or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shewn in the 16th Prop. of this 5th Book.

XIV. *Invertendo*, by inversion; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third. Prop. B. Book 5.

XV. *Componendo*, by composition; when there are four proportionals, and it is inferred that the first, together with the second, is to the second as the third, together with the fourth, is to the fourth. 18th Prop. Book 5.

XVI. *Dividendo*, by division; when there are four proportionals, and it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth. 17th Prop. Book 5.

XVII. *Convertendo*, by conversion; when there are four proportionals, and it is inferred that the first is to its excess above the second as the third to its excess above the fourth. Prop. E. Book 5.

XVIII. *Ex æquali* (sc. *distantia*), or *ex æquo*, from equality of distance; when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes as the first is to the last of the others. Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two.

XIX. *Ex æquali*, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order. And the inference is as mentioned in the preceding definition, whence this is called "ordinate proportion." It is demonstrated in 22d Prop. Book 5.

XX. *Ex æquali in proportione perturbata seu inordinata*, from equality in perturbate or disorderly proportion.* This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank, and so on in a cross order: and the inference is as in the 18th definition. It is demonstrated in the 23d Prop. of Book 5.

AXIOMS.

I. Equimultiples of the same or of equal magnitudes are equal to one another.

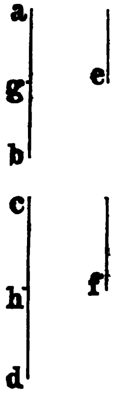
II. Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

III. A multiple of a greater magnitude is greater than the same multiple of a less.

IV. That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROPOSITION I.—THEOREM.

If any number of magnitudes be equimultiples of as many others, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the others.



LET any number of magnitudes ab, cd be equimultiples of as many others e, f, each of each: whatsoever multiple ab is of e, the same multiple shall ab and cd together be of e and f together.

Because ab is the same multiple of e that cd is of f, as many magnitudes as are in ab equal to e, so many are there in cd equal to f. Divide ab into magnitudes equal to e, viz. ag, gb; and cd into ch, hd equal each of them to f: the number therefore of the magnitudes ch, hd, shall be equal to the number of the others ag, gb; and because ag is equal to e, and ch to f, therefore ag and ch together are equal to (i. 2 *ax.*) e and f together. For the same reason, because gb is equal to e, and hd to f; gb and hd together are equal to e and f together. Wherefore as many magnitudes as are in ab equal to e, so many are there in ab, cd together equal to e and f together. Therefore, whatsoever multiple ab is of e, the same multiple is ab and cd together of e and f together.

Therefore, if any magnitudes, how many soever, be equimultiples of as many, each of each, whatsoever multiple any one of them is of its part, the same multiple shall all the first magnitudes be of all the other: for the same demonstration holds in any number of magnitudes, which was here applied to two. Q. E. D.

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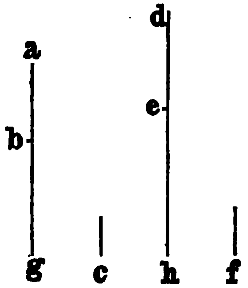
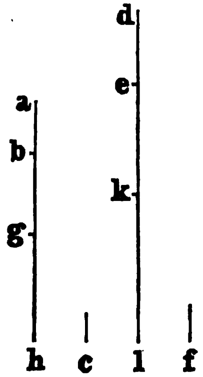
* Archimedes de Sphæra et Cylindro, Prop. 4. lib. 2.

PROPOSITION II.—THEOREM.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

LET **a b** the first, be the same multiple of **c** the second, that **d e** the third is of **f** the fourth; and **b g** the fifth, the same multiple of **c** the second, that **e h** the sixth is of **f** the fourth: then is **a g**, the first, together with the fifth, the same multiple of **c** the second, that **d h**, the third, together with the sixth, is of **f** the fourth.





Because **a b** is the same multiple of **c**, that **d e** is of **f**; there are as many magnitudes in **a b** equal to **c**, as there are in **d e** equal to **f**. In like manner, as many as there are in **b g** equal to **c**, so many are there in **e h** equal to **f**: as many, then, as are in the whole **a g** equal to **c**, so many are there in the whole **d h** equal to **f**; therefore **a g** is the same multiple of **c**, that **d h** is of **f**; that is, **a g** the first and fifth together, is the same multiple of the second **c**, that **d h** the third and sixth together is of the fourth **f**. If, therefore, the first be the same multiple, &c. Q. E. D.

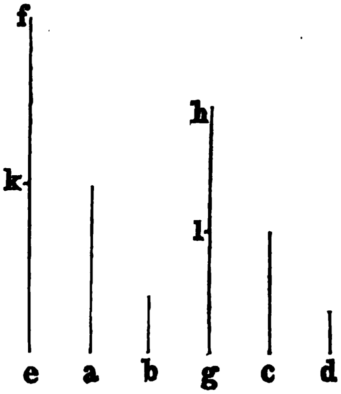
COR. From this it is plain, that if any number of magnitudes **a b**, **b g**, **g h**, be multiples of another **c**; and as many **d e**, **e k**, **k l** be the same multiples of **f**, each of each; the whole of the first, viz. **a h**, is the same multiple of **c**, that the whole of the last, viz. **d l**, is of **f**.

PROPOSITION III.—THEOREM.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

LET **a** the first, be the same multiple of **b** the second, that **c** the third is of **d** the fourth; and of **a**, **c**, let the equimultiples **e f**, **g h** be taken: then **e f** is the same multiple of **b**, that **g h** is of **d**.

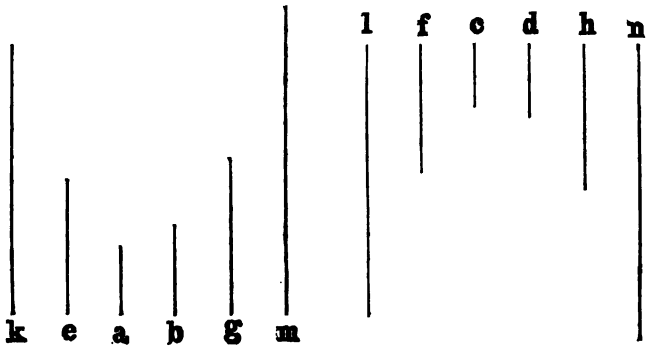
Because **ef** is the same multiple of **a**, that **gh** is of **c**, there are as many magnitudes in **ef** equal to **a**, as are in **gh** equal to **c**. Let **ef** be divided into the magnitudes **ek**, **kf**, each equal to **a**, and **gh** into **gl**, **lh**, each equal to **c**: the number therefore of the magnitudes **ek**, **kf**, shall be equal to the number of the others **gl**, **lh**: and because **a** is the same multiple of **b**, that **c** is of **d**, and that **ek** is equal to **a**, and **gl** to **c**; therefore **ek** is the same multiple of **b**, that **gl** is of **d**. For the same reason, **kf** is the same multiple of **b**, that **lh** is of **d**; and so, if there be more parts in **ef**, **gh**, equal to **a**, **c**: because, therefore, the first **ek** is the same multiple of the second **b**, which the third **gl** is of the fourth **d**, and that the fifth **kf** is the same multiple of the second **b**, which the sixth **lh** is of the fourth **d**; **ef** the first together with the fifth, is the same multiple (ii. 5) of the second **b**, which **gh** the third together with the sixth, is of the fourth **d**. If, therefore, the first, &c. Q. E. D.



Proposition IV.—Theorem.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

Let **a** the first have to **b** the second, the same ratio which the third **c** has to the fourth **d**; and of **a** and **c** let there be taken any equimultiples whatever **e**, **f**; and of **b** and **d** any equimultiples whatever **g**, **h**: then **e** has the same ratio to **g**, which **f** has to **h**.



Take of e and f any equimultiples whatever k, l, and of g, h, any equimultiples whatever m, n: then, because e is the same multiple of a that f is of c: and of e and f have been taken equimultiples k, l; therefore k is the same multiple of a, that l is of c (v. 3). For the same reason, m is the same multiple of b, that n is of d: and because, as a is to b, so is c to d (*hypoth.*), and of a and c have been taken certain equimultiples k, l: and of b and d have been taken certain equimultiples m, n; if therefore k be greater than m, l is greater than n: and if equal, equal; if less, less (v. *def.* 5). And k, l are any equimultiples whatever of e, f; and m, n any whatever of g, h: as therefore e is to g, so is (v. *def.* 5) f to h. Therefore, if the first, &c. Q. E. D.

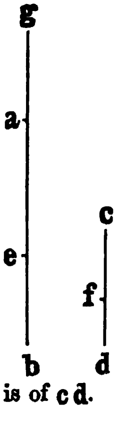
COR. Likewise, if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples whatever of the first and third have the same ratio to the second and fourth. And in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

Let a the first have to b the second, the same ration which the third c has to the fourth d, and of a and c let e and f be any equimultiples whatever; then e is to b, as f to d.

Take of e, f any equimultiples whatever k, l, and of b, d any equimultiples whatever g h; then it may be demonstrated, as before, that k is the same multiple of a, that l is of c. And because a is to b, as c is to d, and of a and c certain equimultiples have been taken, viz. k and l; and of b and d certain equimultiples g, h; therefore, if k be greater than g, l is greater than h; and if equal, equal; if less, less (v. *def.* 5). And k, l are any equimultiples of e, f, and g, h any whatever of b, d; as therefore e is to b, so is f to d. And in the same way the other case is demonstrated.

PROPOSITION V.—THEOREM.

If one magnitude be the same multiple of another which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder that the whole is of the whole.



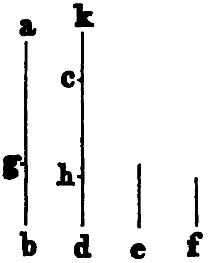
LET the magnitude ab be the same multiple of cd, that ae taken from the first, is of cf taken from the other; the remainder eb shall be the same multiple of the remainder fd, that the whole ab is of the whole cd.

Take ag, the same multiple of fd that ae is of cf: therefore ae is (v. 1) the same multiple of cf, that eg is of cd. But ae, by the hypothesis, is the same multiple of cf, that ab is of cd: therefore eg is the same multiple of cd that ab is of cd; wherefore eg is equal to ab (v. *ax.* 1). Take from them the common magnitude ae; the remainder ag is equal to the remainder eb. Wherefore, since ae is the same multiple of cf, that ag is of fd, and that ag is equal to eb; therefore ae is the same multiple of cf, that eb is of fd. But ae is the same multiple of cf that ab is of cd; therefore eb is the same multiple of fd, that ab is of cd. Therefore, if one magnitude, &c. Q. E. D.

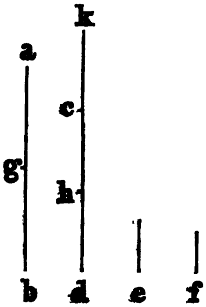
PROPOSITION VI.—THEOREM.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders are either equal to these others, or equimultiples of them.

LET the two magnitudes ab, cd be equimultiples of the two e, f, and ag, ch taken from the first two be equimultiples of the same e, f; the remainders gb, hd are either equal to e, f, or equimultiples of them.



First, let gb be equal to e; hd is equal to f. Make ck equal to f; and because ag is the same multiple of e, that ch is of f, and that gb is equal to e, and ck to f; therefore ab is the same multiple of e that kh is of f. But ab, by the hypothesis, is the same multiple of e that cd is of f; therefore kh is the same multiple of f that cd is of f; wherefore kh is equal to cd ((v. *ax.* 1). Take away the common magnitude ch, then the remainder kc is equal to the remainder hd. But kc is equal to f; hd therefore is equal to f.



But let gb be a multiple of e; then hd is the same multiple of f. Make ck the same multiple of f, that gb is of e. And because ag is the same multiple of e that ch is of f; and gb the same multiple of e that ck is of f; therefore ab is the same multiple of e that kh is of f (v. 2). But ab is the same multiple of e that cd is of f; therefore kh is the same multiple of f that cd is of it: wherefore kh is equal to cd (v. *ax.* 1). Take away ch from both; therefore the remainder kc is equal to the remainder hd. And because gb is the same multiple of e that kc is of f, and that kc is equal to hd; therefore hd is the same multiple of f that gb is of e. If, therefore, two magnitudes, &c. Q. E. D.

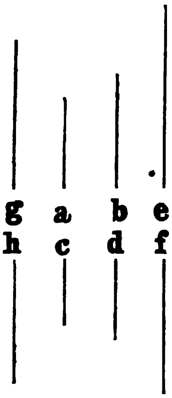
PROPOSITION A.—THEOREM.

If the first of four magnitudes has to the second the same ratio which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

TAKE any equimultiples of each of them, as the doubles of each; then, by def. 5th of this book, if the double of the first be greater than the double of the second, the double of the third is greater than the double of the fourth: but if the first be greater than the second, the double of the first is greater than the double of the second; wherefore also the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth. In like manner, if the first be equal to the second, or less than it, the third can be proved to be equal to the fourth, or less than it. Therefore, if the first, &c. Q. E. D.

PROPOSITION B.—THEOREM.

If four magnitudes are proportionals, they are proportionals also when taken inversely.



IF the magnitude **a** be to **b** as **c** is to **d**, then also inversely **b** is to **a** as **d** to **c**.

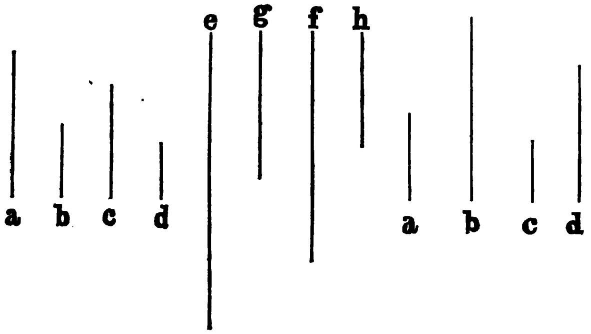
Take of **b** and **d** any equimultiples whatever **e** and **f**; and of **a** and **c** any equimultiples whatever **g** and **h**. First, let **e** be greater than **g**, then **g** is less than **e**; and because **a** is to **b** as **c** is to **d**, and of **a** and **c**, the first and third, **g** and **h** are equimultiples; and of **b** and **d**, the second and fourth, **e** and **f** are equimultiples; and that **g** is less than **e**, **h** is also (v. *def.* 5) less than **f**; that is, **f** is greater than **h**; if therefore **e** be greater than **g**, **f** is greater than **h**. In like manner, if **e** be equal to **g**, **f** may be shewn to be equal to **h**; and if less, less; and **e**, **f** are any equimultiples whatever of **b** and **d**, and **g**, **h** any whatever of **a** and **c**: therefore, as **b** is to **a**, so is **d** to **c**. If then four magnitudes, &c. Q. E. D.

PROPOSITION C.—THEOREM.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third is to the fourth.

LET the first **a** be the same multiple of **b** the second, that **c** the third is of the fourth **d**: **a** is to **b** as **c** is to **d**.

Take of **a** and **c** any equimultiples whatever **e** and **f**; and of **b** and **d** any equimultiples whatever **g** and **h**. Then, because **a** is the same multiple of **b** that **c** is of **d**; and that **e** is the same multiple of **a** that **f** is of **c**: **e** is the same multiple of **b** that **f** is of **d** (v. 3); therefore **e** and **f** are the same multiples of **b** and **d**. But **g** and **h** are equimultiples of **b** and **d**: therefore, if **e** be a greater multiple of **b** than **g** is, **f** is a greater multiple of **d** than **h** is of **d**; that is, if **e** be greater than **g**, **f** is greater



than h. In like manner, if e be equal to g, or less, f is equal to h, or less than it. But e, f are equimultiples, any whatever, of a, c, and g, h any equimultiples whatever of b, d. Therefore a is to b, as c is to d (v. *def.* 5).

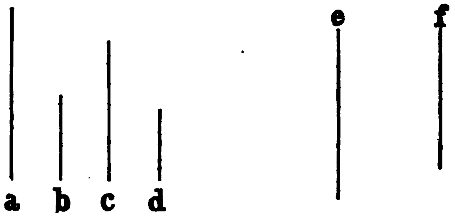
Next, let the first a be the same part of the second b, that the third c is of the fourth d: a is to b as c is to d: for b is the same multiple of a that d is of c: wherefore, by the preceding case, b is to a as d is to c; and inversely (v. B.) a is to b as c is to d. Therefore, if the first be the same multiple, &c. Q. E. D.

PROPOSITION D.—THEOREM.

If the first be to the second as the third to the fourth, and if the first be a multiple, or part, of the second; the third is the same multiple, or the same part, of the fourth.

LET a be to b, as c is to d; and first let a be a multiple of b: c is the same multiple of d.

Take e equal to a, and whatever multiple a or e is of b, make f the same multiple of d. Then, because a is to b as c is to d; and of b the second, and d the fourth, equimultiples have been taken e and f; a is to e, as c to f (v. iv. *cor.*). But a is equal to e, therefore c is equal to f (v. A): and f is the same multiple of d that a is of b. Wherefore c is the same multiple of d that a is of b.

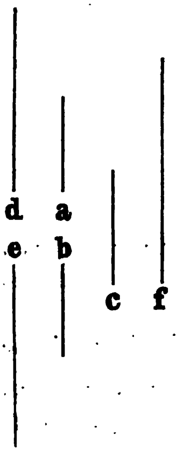


Next, let the first a be a part of the second b; c the third is the same part of the fourth d.

Because a is to b as c is to d; then inversely, b is (v. B.) to a as d to c. But a is a part of b, therefore b is a multiple of a; and, by the preceding case, d is the same multiple of c; that is, c is the same part of d that a is of b. Therefore, if the first, &c. Q. E. D.

Proposition VII.—Theorem.

Equal magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.



Let a and b be equal magnitudes, and c any other. a and b have each of them the same ratio to c, and c has the same ratio to each of the magnitudes a and b.

Take of a and b any equimultiples whatever d and e, and of c any multiple whatever f. Then, because d is the same multiple of a that e is of b, and that a is equal to b; d is (v. *ax.* 1) equal to e. Therefore, if d be greater than f, e is greater than f; and if equal, equal; if less, less. And d, e are any equimultiples of a, b, and f is any multiple of c. Therefore (v. *def.* 5), as a is to c, so is b to c.

Likewise c has the same ratio to a that it has to b: for, having made the same construction, d may in like manner be shewn equal to e. Therefore, if f be greater than d, it is likewise greater than e; and if equal, equal; if less, less. And f is any multiple whatever of c, and d, e are any equimultiples whatever of a, b. Therefore c is to a as c is to b (v. *def.* 5). Therefore, equal magnitudes, &c. Q. E. D.

Proposition VIII.—Theorem.

Of unequal magnitudes the greater has a greater ratio to the same than the less has, and the same magnitude has a greater ratio to the less than it has to the greater.

Let ab, bc be unequal magnitudes, of which ab is the greater, and let d be any magnitude whatever: ab has a greater ratio to d than bc to d; and d has a greater ratio to bc than unto ab.

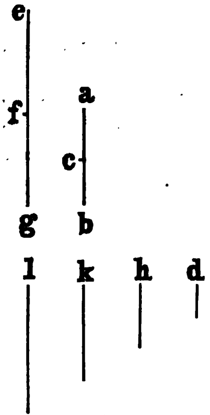


fig. 1.

If the magnitude which is not the greater of the two ac, cb, be not less than d, take ef, fg the doubles of ac, cb, as in fig. 1. But if that which is not the greater of the two ac, cb, be less than d (as in figs. 2 and 3) this magnitude can be multiplied, so as to become greater than d, whether it be ac or cb. Let it be multiplied until it become greater than d, and let the other be multiplied as often; and let ef be the multiple thus taken of ac, and fg the same multiple of cb. Therefore ef and fg are each of them greater than d: and in every one of the cases, take h the double of d, k its triple, and so on, till the multiple of d be that which first becomes greater than fg. Let l be that multiple of d which is first greater than fg, and k the multiple of d which is next less than l.

Then, because l is the multiple of d which is the first that becomes greater than **fg**, the next preceding multiple **k** is not greater than **fg**; that is, **fg** is not less than **k**. And since **ef** is the same multiple of **ac** that **fg** is of **cb**; **fg** is the same multiple of **cb** that **eg** is of **ab** (i. 5); wherefore **eg** and **fg** are equimultiples of **ab** and **cb**. And it was shewn that **fg** was not less than **k**, and, by the construction, **ef** is greater than **d**; therefore the whole **eg** is greater than **k** and **d** together. But **k**, together with **d**, is equal to **l**; therefore **eg** is greater than **l**; but **fg** is not greater than **l**; and **eg**, **fg** are equimultiples of **ab**, **bc**, and **l** is a multiple of **d**; therefore (v. *def.* 7) **ab** has to **d** a greater ratio than **bc** has to **d**.

Also **d** has to **bc** a greater ratio than it has to **ab**. For, having made the same construction, it may be shewn, in like manner, that **l** is greater than **fg**, but that it is not greater than **eg**. And **l** is a multiple of **d**; and **fg**, **eg** are equimultiples of **cb**, **ab**; therefore **d** has to **cb** a greater ratio (v. *def.* 7) than it has to **ab**. Wherefore, of unequal magnitudes, &c. Q. E. D.

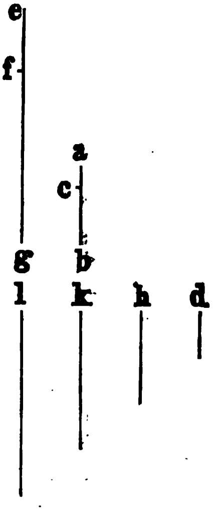


fig. 2.

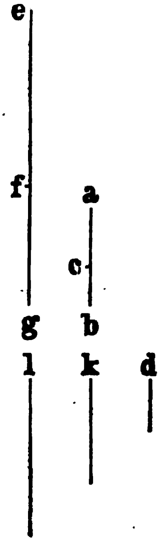
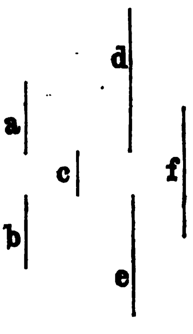


fig. 3.

PROPOSITION IX.—THEOREM.

Magnitudes which have the same ratio to the same magnitude are equal to one another, and those to which the same magnitude has the same ratio are equal to one another.

LET **a**, **b** have each of them the same ratio to **c**; **a** is equal to **b**. For, if they are not equal, one of them is greater than the other. Let **a** be the greater; then, by what was shewn in the preceding proposition, there are some equimultiples of **a** and **b**, and some multiple of **c** such, that the multiple of **a** is greater than the multiple of **c**, but the multiple of **b** is not greater than that of **c**. Let such multiples be taken, and let **d**, **e** be the equimultiples of **a**, **b**, and **f** the multiple of **c**, so that **d** may be greater than **f**, and **e** not greater than **f**. But, because **a** is to **c** as **b** is to **c**, and of **a**, **b** are taken equimultiples **d**, **e**, and of **c** is taken a multiple **f**; and that **d** is greater than **f**; **e** shall also be greater than **f** (v. *def.* 5); but **e** is not greater than **f**; which is impossible; **a** therefore and **b** are not unequal; that is, they are equal.

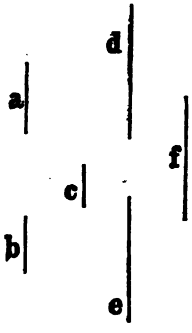


Next, let c have the same ratio to each of the magnitudes a and b; a is equal to b. For, if they are not, one of them is greater than the other; let a be the greater; therefore as was shewn in Prop. 8th, there is some multiple f of c, and some equimultiples e and d, of b and a such, that f is greater than e, and not greater than d; but because c is to b as c is to a, and that f, the multiple of the first, is greater than e, the multiple of the second; f, the multiple of the third, is greater than d, the multiple of the fourth (v. *def.* 5). But f is not greater than d, which is impossible. Therefore a is equal to b. Wherefore, magnitudes, &c. Q. E. D.

Proposition X.—Theorem.

That magnitude which has a greater ratio than another has unto the same magnitude is the greater of the two. And that magnitude to which the same has a greater ratio than it has unto another magnitude is the lesser of the two.

Let a have to c a greater ratio than b has to c; a is greater than b: for because a has a greater ratio to c than b has to c, there are (v. *def.* 7) some equimultiples of a and b, and some multiple of c, such that the multiple of a is greater than the multiple of c, but the multiple of b is not greater than it: let them be taken, and let d, e be equimultiples of a, b, and f a multiple of c, such that d is greater than f, but e is not greater than f: therefore d is greater than e: and because d and e are equimultiples of a and b, and d is greater than e; therefore a is (v. 4 *ax.*) greater than b.



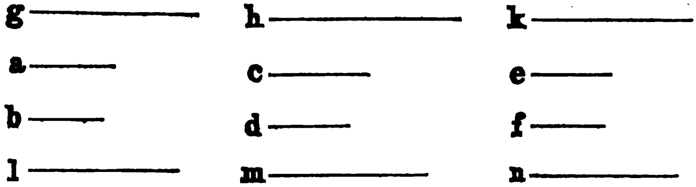
Next, let c have a greater ratio to b than it has to a; b is less than a: for (v. *def.* 7) there is some multiple f of c, and some equimultiples e and d of b and a, such that f is greater than e, but is not greater than d; e therefore is less than d; and because e and d are equimultiples of b and a, therefore b is (v. 4 *ax.*) less than a. That magnitude, therefore, &c. Q. E. D.

Proposition XI.—Theorem.

Ratios that are the same to the same ratio are the same to one another.

Let a be to b as c is to d; and as c to d so let e be to f; a is to b as e to f.

Take of a, c, e, any equimultiples whatever g, h, k; and of b, d, f, any equimultiples whatever l, m, n. Therefore, since a is to b, as c to d, and g, h are taken equimultiples of a, c, and l, m, of b, d; if g be greater than l, h is greater than m; and if equal, equal; and if less, less (v. *def.* 5). Again, because c is to d as e is to f, and h, k are taken equimultiples of c, e; and m, n, of d, f; if h be greater than m, k is greater than n; and if equal, equal; and if less, less: but if g be greater than l, it has been shewn that **h** is greater than **m** : and if equal, equal; and if less, less; therefore if **g** be greater than **l**, **k** is greater than **n**; and if equal, equal; and if less, less: and **g**, **k** are any equimultiples whatever of **a**, **e**; and **l**, **n** any whatever of **b**, **f**: therefore, as **a** is to **b**, so is **e** to **f** (v. *def.* 5). Wherefore, ratios, &c. Q. E. D.

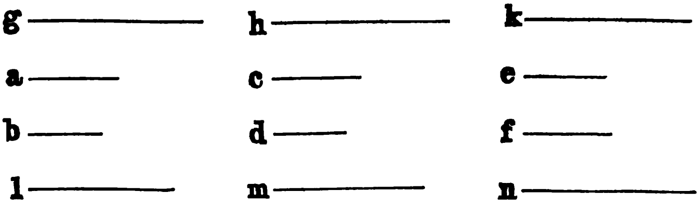


Proposition XII.—Theorem.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any number of magnitudes **a**, **b**, **c**, **d**, **e**, **f** be proportionals; that is, as **a** is to **b**, so is **c** to **d**, and **e** to **f**. As **a** is to **b**, so shall **a**, **c**, **e** together be to **b**, **d**, **f** together.

Take of **a**, **c**, **e** any equimultiples whatever **g**, **h**, **k**; and of **b**, **d**, **f** any equimultiples whatever **l**, **m**, **n** : then, because **a** is to **b**, as **c** is to **d**, and as **e** to **f**; and that **g**, **h**, **k** are equimultiples of **a**, **c**, **e**, and **l**, **m**, **n**, equimultiples of **b**, **d**, **f**; if **g** be greater than **l**, **h** is greater than **m**, and **k** greater than **n**; and if equal, equal; and if less, less (v. *def.* 5). Wherefore, if **g** be greater than **l**, then **g**, **h**, **k** together are greater than **l**, **m**, **n** together; and if equal, equal; and if less, less. And **g**, and **g**, **h**, **k**, together are any equimultiples of **a**, and **a**, **c**, **e** together; because if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the same multiple is the whole of the whole (v. 1). For the same reason **l**, and **l**, **m**, **n** are any equimultiples of **b**, and **b**, **d**, **f**: as therefore **a** is to **b**, so are **a**, **c**, **e**, together to **b**, **d**, **f** together. Wherefore, if any number, &c. Q. E. D.

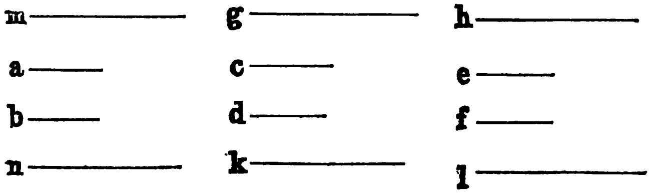


PROPOSITION XIII.—THEOREM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.

LET a the first, have the same ratio to b the second, which c the third, has to d the fourth, but c the third, to d the fourth, a greater ratio than e the fifth, to f the sixth : also the first a shall have to the second b a greater ratio than the fifth e to the sixth f.

Because c has a greater ratio to d, than e to f, there are some equimultiples of c and e, and some of d and f, such that the multiple of c is greater than the multiple of d, but the multiple of e is not greater than the multiple of f (v. *def.* 7) : let such be taken, and of c, e, let g, h be equimultiples, and k, l equimultiples of d, f, so that g be greater than k, but h not greater than l ; and whatever multiple g is of c, take m the same multiple of a ; and whatever multiple k is of d, take n the same multiple of b : then, because a is to b, as c to d, and of a and c, m and g are equimultiples : and of b and d, n and k are equimultiples ; if m be greater than n, g is greater than k ; and if equal, equal ; and if less, less (v. *def.* 5) ; but g is greater than k ; therefore m is greater than n : but h is not greater than l ; and m, h are equimultiples of a, e ; and n, l equimultiples of b f : therefore a has a greater ratio to b than e has to f (v. *def.* 7). Wherefore, if the first, &c. Q. E. D.



COR. And if the first have a greater ratio to the second, than the third has to the fourth, but the third the same ratio to the fourth, which the fifth has to the sixth ; it may be demonstrated, in like manner, that the first has a greater ratio to the second, than the fifth has to the sixth.

PROPOSITION XIV.—THEOREM.

If the first has to the second the same ratio which the third has to the fourth ; then, if the first be greater than the third, the second shall be greater than the fourth ; and if equal, equal ; and if less, less.

LET the first a have to the second b, the same ratio which the third c, has to the fourth d ; if a be greater than c, b is greater than d.

Because a is greater than c, and b is any other magnitude, a has to b a greater ratio than c to b (v. 8) : but, as a is to b so is c to d ; therefore also c has to d a greater ratio than c has to b (v. 13). But of two magnitudes, that to which the same has the greater ratio is the lesser (v. 10). Wherefore **d** is less than **b**; that is, **b** is greater than **d**.

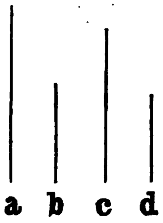


fig. 1.

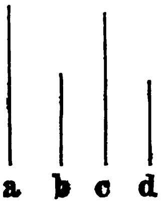


fig. 2.

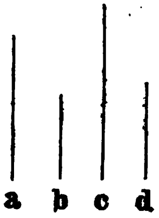


fig. 3.

Secondly, if **a** be equal to **c**, **b** is equal to **d**: for **a** is to **b** as **c**, that is **a**, is to **d**: **b** therefore is equal to **d** (v. 9).

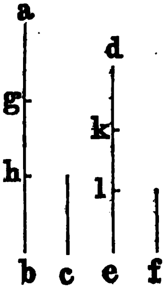
Thirdly, if **a** be less than **c**, **b** shall be less than **d**: for **c** is greater than **a**, and because **c** is to **d** as **a** is to **b**, **d** is greater than **b**, by the first case; wherefore **b** is less than **d**. Therefore if the first, &c. Q. E. D.

Proposition XV.—Theorem.

Magnitudes have the same ratio to one another which their equimultiples have.

Let **ab** be the same multiple of **c** that **de** is of **f**; **c** is to **f** as **ab** to **de**.

Because **ab** is the same multiple of **c** that **de** is of **f**; there are as many magnitudes in **ab** equal to **c**, as there are in **de** equal to **f**: let **ab** be divided into magnitudes, each equal to **c**, viz. **ag**, **gh**, **hb**; and **de** into magnitudes, each equal to **f**, viz. **dk**, **kl**, **le**: then the number of the first **ag**, **gh**, **hb**, shall be equal to the number of the last **dk**, **kl**, **le**: and because **ag**, **gh**, **hb** are all equal, and that **dk**, **kl**, **le**, are also equal to one another; therefore **ag** is to **dk** as **gh** to **kl**, and as **hb** to **le** (v. 7): and as one of the antecedents to its consequent, so are all the antecedents together to all the consequents together (v. 12); wherefore, as **ag** is to **dk**, so is **ab** to **de**: but **ag** is equal to **c**, and **dk** to **f**: therefore, as **c** is to **f** so is **ab** to **de**. Therefore magnitudes, &c. Q. E. D.

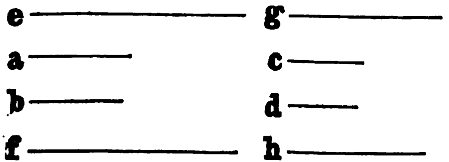


PROPOSITION XVI.—THEOREM.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

LET the four magnitudes **a**, **b**, **c**, **d** be proportionals, viz. as **a** is to **b**, so is **c** to **d**: they shall also be proportionals when taken alternately; that is, **a** is to **c**, as **b** to **d**.

Take of **a** and **b** any equimultiples whatever **e** and **f**; and of **c** and **d** take any equimultiples whatever **g** and **h**: and because **e** is the same multiple of **a** that **f** is of **b**, and that magnitudes have the same ratio to one another which their equimultiples have (v. 15); therefore **a** is to **b**, as **e** is to **f**; but as **a** is to **b**, so is **c** to **d**; wherefore as **c** is to **d**, so (v. 11) is **e** to **f**: again, because **g**, **h** are equimultiples of **c**, **d**, as **c** is to **d** so is **g** to **h** (v. 15): but as **c** is to **d**, so is **e** to **f**. Wherefore, as **e** is to **f** so is **g** to **h** (v. 11). But when four magnitudes are proportionals, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; if less, less (v. 14). Wherefore, if **e** be greater than **g**, **f** likewise is greater than **h**; and if equal, equal; if less, less; and **e**, **f** are any equimultiples whatever of **a**, **b**; and **g**, **h** any whatever of **c**, **d**. Therefore **a** is to **c** as **b** to **d** (v. *def.* 5). If then four magnitudes, &c. Q. E. D.

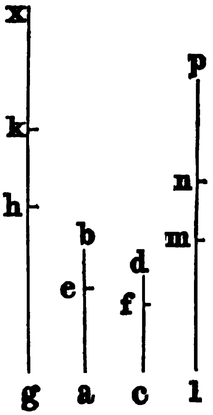


PROPOSITION XVII.—THEOREM.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

LET **ab**, **be**, **cd**, **df** be the magnitudes taken jointly which are proportionals; that is, as **ab** to **be** so is **cd** to **df**: they shall also be proportionals taken separately, viz. as **ae** to **eb**, so is **cf** to **fd**.

Take of **ae**, **eb**, **cf**, **fd** any equimultiples whatever **gh**, **hk**, **lm**, **mn**: and again, of **eb**, **fd** take any equimultiples whatever **kx**, **np**: and because **gh** is the same multiple of **ae** that **hk** is of **eb**, wherefore **gh** is the same multiple (v. 1) of **ae** that **gk** is of **ab**: but **gh** is the same multiple of **ae** that **lm** is of **cf**; wherefore **gk** is the same multiple of **ab** that **lm** is of **cf**. Again, because **lm** is the same multiple of **cf** that **mn** is of **fd**; therefore **lm** is the same multiple (v. 1) of **cf** that **ln** is of **cd**: but **lm** was shewn to be the same multiple of **cf** that **gk** is of **ab**; **gk** therefore is the same multiple of **ab** that **ln** is of **cd**; that is, **gk**, **ln** are equimultiples of **ab**, **cd**. Next, because **hk** is the same multiple of **eb** that **mn** is of **fd**; and that **kx** is also the same multiple of **eb** that **np** is of **fd**; therefore **hx** is the same multiple (v. 2) of **e b**, that **m p** is of **f d**. And because **a b** is to **b e** as **c d** is to **d f**, and that of **a b** and **c d**, **g k** and **l n** are equimultiples, and of **e b** and **f d**, **h x** and **m p** are equimultiples; if **g k** be greater than **h x**, then **l n** is greater than **m p**; and if equal, equal; and if less, less (v *def.* 5); but if **g h** be greater than **k x**, by adding the common part **h k** to both, **g k** is greater than **h x**; wherefore also **l n** is greater than **m p**; and by taking away **m n** from both, **l m** is greater than **n p**: therefore, if **g h** be greater than **k x**, **l m** is greater than **n p**. In like manner it may be demonstrated, that if **g h** be equal to **k x**, **l m** likewise is equal to **n p**; and if less, less: and **g h**, **l m** are any equimultiples whatever of **a e**, **c f**, and **k x**, **n p** are any whatever of **e b**, **f d**. Therefore (v. *def.* 5), as **a e** is to **e b**, so is **c f** to **f d**. If then magnitudes, &c. Q. E. D.



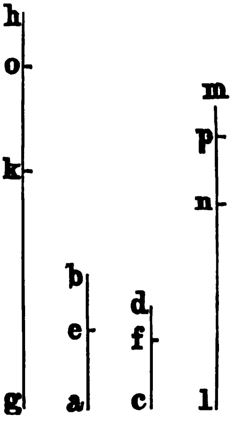
PROPOSITION XVIII.—THEOREM.

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly, that is, if the first be to the second as the third to the fourth, the first and second together shall be to the second as the third and fourth together to the fourth.

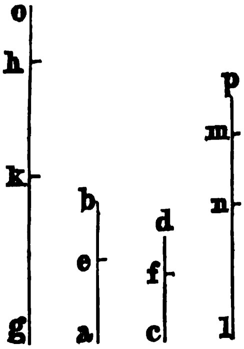
LET **a e**, **e b**, **c f**, **f d** be proportionals; that is, as **a e** to **e b**, so is **c f** to **f d**; they shall also be proportionals when taken jointly; that is, as **a b** to **b e**, so **c d** to **d f**.

Take of **a b**, **b e**, **c d**, **d f** any equimultiples whatever **g h**, **h k**, **l m**, **m n**: and again, of **b e**, **d f**, take any equimultiples whatever **k o**, **n p**: and because **k o**, **n p** are equimultiples of **b e**, **d f**; and that **k h**, **n m** are equimultiples likewise of **b e**, **d f**, if **k o**, the multiple of **b e**, be greater than **k h**, which is a multiple of the same **b e**, **n p** likewise the multiple of **d f**, shall be greater than **m n**, the multiple of the same **d f**; and if **k o** be equal to **k h**, **n p** shall be equal to **n m**; and if less, less.

First, let **k o** not be greater than **k h**, therefore **n p** is not greater than **n m**: and because **g h**, **h k**, are equimultiples of **a b**, **b e**, and that **a b** is greater than **b e**, therefore **g h** is greater (v. *ax.* 3) than **h k**; but **k o** is not greater than **k h**, wherefore **g h** is greater than **k o**. In like manner it may be shewn, that **l m** is greater than **n p**. Therefore, if **k o** be not greater than **k h**, then **g h**, the multiple of **a b**, is always greater than **k o**, the multiple of **b e**; and likewise **l m**, the multiple of **c d**, greater than **n p**, the multiple of **d f**.

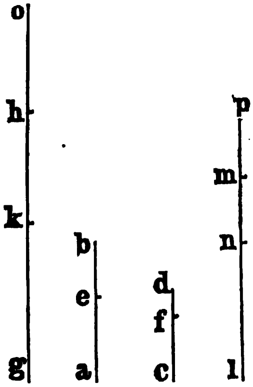


Next, let **k o** be greater than **k h**: therefore, as has been shewn, **n p** is greater than **n m** : and because the whole **g h** is the same multiple of the whole **a b**, that **h k** is of **b e**, the remainder **g k** is the same multiple of the remainder **a e** that **g h** is of **a b** (v. 5) : which is the same that **l m** is of **c d**. In like manner, because **l m** is the same multiple of **c d**, that **m n** is of **d f**, the remainder **l n** is the same multiple of the remainder **c f**, that the whole **l m** is of the whole **c d** (v. 5) : but it was shewn that **l m** is the same multiple of **c d**, that **g k** is of **a e** ; therefore **g k** is the same multiple of **a e**, that **l n** is of **c f** ; that is, **g k**, **l n** are equimultiples of **a e**, **c f** : and because **k o**, **n p**, are equimultiples of **b e**, **d f**, if from **k o**, **n p**, there be taken **k h**, **n m**, which are likewise equimultiples of **b e**, **d f**, the remainders **h o**, **m p** are either equal to **b e**, **d f**, or equimultiples of them (v. 6).



First, let **h o**, **m p** be equal to **b e**, **d f** ; and because **a e** is to **e b**, as **c f** to **f d**, and that **g k**, **l n** are equimultiples of **a e**, **c f** ; **g k** shall be to **e b**, as **l n** to **f d** (v. 4 *cor.*) : but **h o** is equal to **e b**, and **m p** to **f d** ; wherefore **g k** is to **h o**, as **l n** to **m p**. If therefore **g k** be greater than **h o**, **l n** is greater than **m p** ; and if equal, equal ; and if less, less (5 *ax.*).

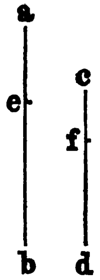
But let **h o**, **m p** be equimultiples of **e b**, **f d** ; and because **a e** is to **e b**, as **c f** to **f d**, and that of e, **c f** are taken equimultiples **g k**, **l n** ; and of **e b**, **f d**, the equimultiples **h o**, **m p** ; if **g k** be greater than **h o**, **l n** is greater than **m p** ; and if equal, equal ; and if less, less (v. *def.* 5) ; which was likewise shewn in the preceding case. If therefore **g h** be greater than **k o**, taking **k h** from both, **g k** is greater than **h o** ; wherefore also **l n** is greater than **m p** ; and consequently adding **n m** to both, **l m** is greater than **n p** : therefore, if **g h** be greater than **k o**, **l m** is greater than **n p**. In like manner it may be shewn, that if **g h** be equal to **k o**, **l m** is equal to **n p** ; and if less, less. And in the case in which **k o** is not greater than **k h**, it has been shewn that **g h** is always greater than **k o**, and likewise **l m** than **n p** : but **g h**, **l m** are any equimultiples of **a b**, **c d**, and **k o**, **n p** are any whatever of **b e**, **d f** ; therefore (v. *def.* 5), as **a b** is to **b e**, so is **c d** to **d f**. If then magnitudes, &c. Q. E. D.



Proposition XIX.—Theorem.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.

Let the whole **ab** be to the whole **cd**, as **ae**, a magnitude taken from **ab**, to **cf**, a magnitude taken from **cd**; the remainder **eb** shall be to the remainder **fd**, as the whole **ab** to the whole **cd**.

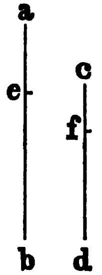


Because **ab** is to **cd**, as **ae** to **cf**; likewise, alternately (v. 16), **ba** is to **ae**, as **dc** to **cf**; and because if magnitudes, taken jointly, be proportionals, they are also proportionals (v. 17) when taken separately; therefore, as **be** is to **ea**, so is **df** to **fc**, and alternately, as **be** is to **df**, so is **ea** to **fc**; but, as **ae** to **cf**, so, by the hypothesis, is **ab** to **cd**.: therefore also **be**, the remainder, shall be to the remainder **df** as the whole **ab** to the whole **cd**. Wherefore, if the whole, &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other, the remainder likewise is to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

Proposition E.—Theorem.

If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.

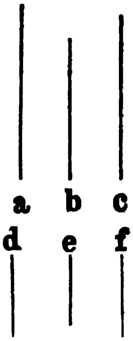


Let **ab** be to **be**, as **cd** to **df**; then **ba** is to **ae**, as **dc** to **cf**.

Because **ab** is to **be**, as **cd** to **df**, by division (v. 17), **ae** is to **eb**, as **cf** to **fd**; and by inversion (v. B.), **be** is to **ea** as **df** to **fc**. Wherefore, by composition (v. 18), **ba** is to **ae** as **dc** is to **cf**. If, therefore, four magnitudes, &c. Q. E. D.

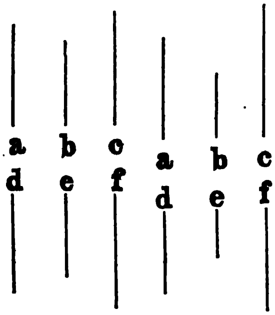
PROPOSITION XX.—THEOREM.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth, and if equal, equal; and if less, less.



LET **a**, **b**, **c** be three magnitudes, and **d**, **e**, **f** other three, which, taken two and two, have the same ratio, viz. as **a** is to **b**, so is **d** to **e**; and as **b** to **c**, so is **e** to **f**. If **a** be greater than **c**, **d** shall be greater than **f**; and if equal, equal; and if less, less.

Because **a** is greater than **c**, and **b** is any other magnitude, and that the greater has to the same magnitude a greater ratio than the less has to it (v. 8); therefore **a** has to **b** a greater ratio than **c** has to **b**; but as **d** is to **e**, so is **a** to **b**; therefore (v. 13) **d** has to **e** a greater ratio than **c** to **b**; and because **b** is to **c**, as **e** to **f**, by inversion, **c** is to **b**, as **f** is to **e**; and **d** was shewn to have to **e** a greater ratio than **c** to **b**; therefore **d** has to **e** a greater ratio than **f** to **e** (v. 13. *cor.*). But the magnitude which has a greater ratio than another to the same magnitude, is the greater of the two (v. 10); **d** is therefore greater than **f**.



Secondly, let **a** be equal to **c**; **d** shall be equal to **f**. Because **a** and **c** are equal to one another, **a** is to **b** as **c** is to **b** (v. 7): but **a** is to **b** as **d** to **e**; and **c** is to **b** as **f** to **e**; wherefore **d** is to **e** as **f** to **e** (v. 11); and therefore **d** is equal to **f** (v. 9).

Next, let **a** be less than **c**; **d** shall be less than **f**; for **c** is greater than **a**, and, as was shewn in the first case, **c** is to **b**, as **f** to **e**, and in like manner, **b** is to **a**, as **e** to **d**; therefore **f** is greater than **d**, by the first case; and therefore **d** is less than **f**. Therefore, if there be three magnitudes, &c. Q. E. D.

PROPOSITION XXI.—THEOREM.

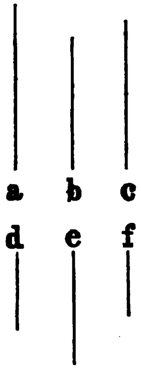
If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

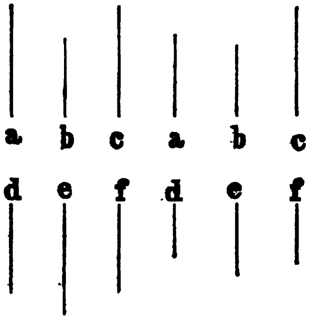
LET **a**, **b**, **c** be three magnitudes, and **d**, **e**, **f** other three, which have the same ratio, taken two and two, but in a cross order, viz. as **a** is to **b**, so is **e** to **f**, and as **b** is to **c**, so is **d** to **e**. If **a** be greater than **c**, **d** shall be greater than **f**; and if equal, equal; and if less, less.

Because **a** is greater than **c**, and **b** is any other magnitude, **a** has to **b** a greater ratio (v. 8) than **c** has to **b**: but as **e** is to **f**, so is **a** to **b**:

therefore (v. 13) e has to f a greater ratio than c to b. And because b is to c as d to e, by inversion, c is to b as e to d: and e was shewn to have to f a greater ratio than c to b; therefore e has to f a greater ratio than e to d (v. 13. *cor.*); but the magnitude to which the same has a greater ratio than it has to another, is the lesser of the two (v. 10): f therefore is less than d; that is, d is greater than f.

Secondly, let a be equal to c; d shall be equal to f. Because a and c are equal, a is (v. 7) to b, as c is to b: but a is to b, as e to f; and c is to b, as e to d; wherefore e is to f, as e to d (v. 11); and therefore d is equal to f (v. 9).



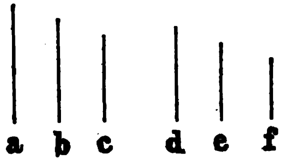


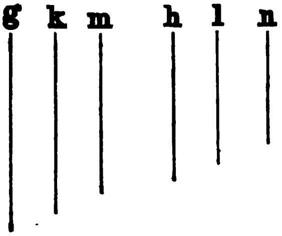
Next, let a be less than c: d shall be less than f: for c is greater than a, and, as was shewn, c is to b, as e to d, and in like manner b is to a, as f to e; therefore f is greater than d, by case first; and therefore, d is less than f. Therefore, if there be three magnitudes, &c. Q. E. D.

Proposition XXII.—Theorem.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last. N.B.—This is usually cited by the words "ex æquali," or "ex æquo."

First, let there be three magnitudes a, b, c, and as many others d, e, f,





which, taken two and two, have the same ratio : that is, such that a is to b as d to e ; and as b is to c, so is e to f ; a shall be to c, as d to f.

Take of a and d any equimultiples whatever g and h ; and of b and e any equimultiples whatever k and l ; and of c and f any whatever m and n : then, because a is to b as d to e, and that g, h are equimultiples of a, d, and k, l equimultiples of b, e ; as g is to k, so is (v. 4) h to l. For the same reason, k is to m, as l is to n ; and because there are three magnitudes g, k, m, and other three h l, n, which, two and two, have the same ratio ; if g be greater than m, h is greater than n ; and if equal, equal ; and if less, less (v. 20) ; and g, h are any equimultiples whatever of a, d, and m, n are any equimultiples whatever of c f : therefore (v. *def.* 5), as a is to c, so is d to f.

a, b, c, d,
e, f, g, h,

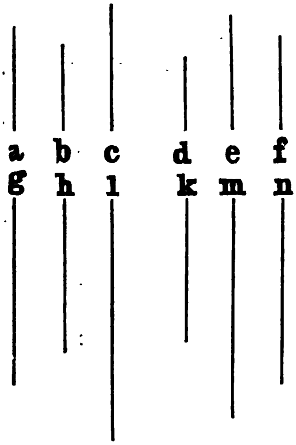
Next, let there be four magnitudes a, b, c, d, and other four e, f, g, h, which, two and two, have the same ratio, viz. as a is to b, so is e to f ; and as b to c, so f to g ; and as c to d, so g to h : a shall be to d, as e to h.

Because a, b, c, are three magnitudes, and e, f, g, other three, which, taken two and two, have the same ratio ; by the foregoing case, a is to c, as e to g : but c is to d, as g is to h ; wherefore again, by the first case, a is to d, as e to h ; and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

Proposition XXIII.—Theorem.

If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio ; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last. N.B. This is usually cited by the words "ex æquali in proportione perturbata," or "ex æquo perturbato."

First, let there be three magnitudes a, b, c, and other three, d, e, f which, taken two and two, in a cross order, have the same ratio : that is, such that a is to b as e to f ; and as b is to c, so is d to e : a is to c, as d to f.



Take of a, b, d any equimultiples whatever g, h, k ; and of c, e, f any equimultiples whatever l, m, n : and because g, h are equimultiples of a b, and that magnitudes have the same ratio which their equimultiples have (v. 15) : as a is to b, so is g to h : and for the same reason, as e is to f, so is m to n : but as a is to b, so is e to f ; therefore g is to h, as m is to n (v. 11). And because as b is to c, so is d to e, and that h, k are equimultiples of b, d, and l, m of c, e ; as h is to l, so is (v. 4) k to m : and it has been shewn that g is to h, as m to n : then, because there are three magnitudes g, h, l and other three k, m, n, which have the same ratio taken two and two in a cross order; if **g** be greater than **l**, **k** is greater than **n**: and if equal, equal; and if less, less (v. 21); and **g**, **k** are any equimultiples whatever of **a**, **d**; and **l**, **n** any whatever of **c**, **f**; as therefore **a** is to **c**, so is **d** to **f**.

Next, let there be four magnitudes **a**, **b**, **c**, **d**, and other four **e**, **f**, **g**, **h**, which, taken two and two in a cross order, have the same ratio, viz. **a** to **b** as **g** to **h**; **b** to **c** as **f** to **g**; and **c** to **d** as **e** to **f**. **a** is to **d** as **e** to **h**.

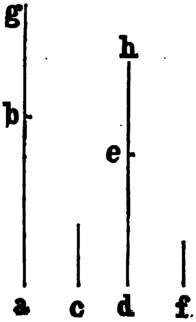
a, **b**, **c**, **d**,
e, **f**, **g**, **h**,

Because **a**, **b**, **c** are three magnitudes, and **f**, **g**, **h** other three, which, taken two and two in a cross order, have the same ratio; by the first case, **a** is to **c**, as **f** to **h**; but **c** is to **d**, as **e** is to **f**: wherefore again, by the first case, **a** is to **d**, as **e** to **h**: and so on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

Proposition XXIV.—Theorem.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Let **ab** the first, have to **c** the second, the same ratio which **de** the third has to **f** the fourth; and let **bg** the fifth have to **c** the second, the same ratio which **eh** the sixth has to **f** the fourth; **ag**, the first and fifth together, shall have to **c** the second, the same ratio which **dh**, the third and sixth together, has to **f** the fourth.



Because **bg** is to **c**, as **eh** to **f**; by inversion, **c** is to **bg**, as **f** to **eh**: and because, as **ab** is to **c**, so is **de** to **f**: and as **c** to **bg**, so **f** to **eh**; *ex æquali* (v. 22), **ab** is to **bg**, as **de** to **eh**: and because these magnitudes are proportionals, they shall likewise be proportionals when taken jointly (v. 18); as therefore **ag** is to **gb**, so is **dh** to **he**: but as **gb** to **c**, so is **he** to **f**. Therefore *ex æquali* (v. 22), as **ag** is to **c**, so is **dh** to **f**. Wherefore, if the first, &c. Q. E. D.

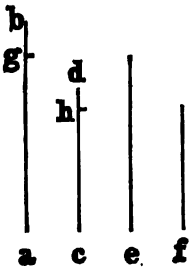
Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

PROPOSITION XXV.—THEOREM.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

LET the four magnitudes ab, cd, e, f be proportionals, viz. ab to cd, as e to f; and let ab be the greatest of them, and consequently f the least (v. 14 and A.), ab, together with f, are greater than cd, together with e.



Take ag equal to e, and ch equal to f: then, because as ab is to cd, so is e to f; and that ag is equal to e, and ch equal to f, ab is to cd, as ag to ch. And because ab the whole, is to the whole cd, as ag is to ch, likewise the remainder gb shall be to the remainder hd, as the whole ab is to the whole cd (v. 19): but ab is greater than cd, therefore (v. A.) gb is greater than hd: and because ag is equal to e, and ch to f; ag and f together are equal to ch and e together. If therefore to the unequal magnitudes gb, hd, of which gb is the greater, there be added equal magnitudes, viz. to gb the two ag and f, and ch and e to hd; ab and f together are greater than cd and e. Therefore, if four magnitudes, &c. Q. E. D.

PROPOSITION F.—THEOREM.

Ratios which are compounded of the same ratios are the same with one another.

LET a be to b as d to e; and b to c as e to f: the ratio which is compounded of the ratios of a to b, and b to c, which by the definition of compound ratio, is the ratio of a to c, is the same with the ratio of d to f, which, by the same definition, is compounded of the ratios of d to e, and e to f.

a	b	c
d	e	f

Because there are three magnitudes a, b, c, and three others d, e, f, which, taken two and two in order, have the same ratio: *ex æquali* a is to c, as d to f (v. 22).

Next, let a be to b, as e to f, and b to c, as d to e; therefore, *ex æquali in proportione perturbata* (v. 23), a is to c, as d to f; that is, the ratio of a to c, which is compounded of the ratios of a to b, and b to c, is the same with the ratio of d to f, which is compounded of the ratios of d to e, and e to f. And in like manner the proposition may be demonstrated, whatever be the number of ratios in either case.

a	b	c
d	e	f

PROPOSITION G.—THEOREM.

If several ratios be the same with several ratios, each to each, the ratio which is compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.

LET **a** be to **b** as **e** to **f**; and **c** to **d** as **g** to **h**: and let **a** be to **b** as **k** to **l**; and **c** to **d** as **l** to **m**; then the ratio of **k** to **m**, by the definition of compound ratio, is compounded of the ratios of **k** to **l**, and **l** to **m**, which are the same with the ratios of **a** to **b**, and **c** to **d**; and as **e** to **f**, so let **n** be to **o**; and as **g** to **h**, so let **o** be to **p**; then the ratio of **n** to **p** is compounded of the ratios of **n** to **o**, and **o** to **p**, which are the same with the ratios of **e** to **f**, and **g** to **h**: and it is to be shewn that the ratio of **k** to **m** is the same with the ratio of **n** to **p**, or that **k** is to **m** as **n** to **p**.

a **b** **c** **d** **k** **l** **m**
e **f** **g** **h** **n** **o** **p**

Because **k** is to **l** as (**a** to **b**, that is, as **e** to **f**, that is, as) **n** to **o**; and as **l** to **m**, so is (**c** to **d**, and so is **g** to **h**, and so is) **o** to **p**: *ex æquali* (v. 22) **k** is to **m** as **n** to **p**. Therefore, if several ratios, &c. Q. E. D.

PROPOSITION H.—THEOREM.

If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first, be the same with one of the last ratios, or with the ratio compounded of any of the last; then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.

LET the first ratios be those of **a** to **b**, **b** to **c**, **c** to **d**, **d** to **e**, and **e** to **f**; and let the other ratios be those of **g** to **h**, **h** to **k**, **k** to **l**, and **l** to **m**; also, let the ratio of **a** to **f**, which is compounded of (*def. of compounded ratio*) the first ratios, be the same with the ratio of **g** to **m**, which is compounded of the other ratios; and besides, let the ratio of **a** to **d**, which is compounded of the ratios of **a** to **b**, **b** to **c**, **c** to **d**, be the same with the ratio of **g** to **k**, which is compounded of the ratios of **g** to **h**, and **h** to **k**: then the ratio compounded of the remaining first ratios, to wit, of the ratios of **d** to **e**, and **e** to **f**, which compounded ratio is the ratio of **d** to **f**, is the same with the ratio of **k** to **m**, which is compounded of the remaining ratios of **k** to **l**, and **l** to **m** of the other ratios.

a **b** **c** **d** **e** **f**
g **h** **k** **l** **m**

Because, by the hypothesis, **a** is to **d**, as **g** to **k**, by inversion (v. B.), **d** is to **a**, as **k** to **g**; and as **a** is to **f**, so is **g** to **m**; therefore (v. 22), *ex æquali*, **d** is to **f**, as **k** to **m**. If therefore a ratio which is, &c. Q. E. D.

PROPOSITION K.—THEOREM.

If there be any number of ratios, and any number of other ratios, such that the ratio compounded of ratios which are the same with the first ratios, each to each, is the same with the ratio compounded of ratios which are the same, each to each, with the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same with several of the first ratios, each to each, be the same with one of the last ratios, or with the ratio compounded of ratios which are the same, each to each, with several of the last ratios; then the ratio compounded of ratios which are the same with the remaining ratios of the first, each to each, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of ratios which are the same with those remaining of the last, each to each, or with the remaining ratio of the last.

LET the ratios of a to b, c to d, e to f, be the first ratios: and the ratios of g to h, k to l, m to n, o to p, q to r, be the other ratios. And let a be to b, as s to t; and c to d, as t to v, and e to f, as v to x. Therefore, by the definition of compound ratio, the ratio of s to x is compounded of the ratios of s to t, t to v, and v to x, which are the same with the ratios of a to b, c to d, e to f, each to each. Also, as g to h, so let y be to z; and k to l as z to *a*, m to n as *a* to *b*, o to p as *b* to *c*; and q to r as *c* to *d*. Therefore by the same definition, the ratio of y to *d* is compounded of the ratios of y to z, z to *a*, *a* to *b*, *b* to *c*, and *c* to *d*, which are the same, each to each, with the ratios of g to h, k to l, m to n, o to p, and q to r. Therefore, by the hypothesis, s is to x as y to *d*. Also, let the ratio of a to b, that is, the ratio of s to t, which is one of the first ratios, be the same with the ratio of *e* to *g*, which is compounded of the ratios of *e* to *f*, and *f* to *g*, which, by the hypothesis, are the same with the ratios of g to h and k to l, two of the other ratios; and let the ratio of *h* to *l* be that which is compounded of the ratios of *h* to *k*, and *k* to *l*, which are the same with the remaining first ratios, viz. of c to d, and e to f; also, let the ratio of *m* to *p* be that which is compounded of the ratios of *m* to *n*, *n* to *o*, and *o* to *p*, which are the same, each to each, with the remaining other ratios, viz. of m to n, o to p, and q to r. Then the ratio of *h* to *l* is the same with the ratio of *m* to *p*, or *h* is to *l* as *m* to *p*.

h, *k*, *l*.
a, b; c, d; e, f.   s, t, v, x.
g, h; k, l; m, n; o, p; q, r.   y, z, *a*, *b*, *c*, *d*.
e, *f*, *g*.   *m*, *n*, *o*, *p*.

h, *k*, *l*.
a, b; c, d; e, f.   s, t, v, x.
g, h; k, l; m, n; o, p; q, r.   y, z, *a*, *b*, *c*, *d*.
e, *f*, *g*.   *m*, *n*, *o*, *p*.

Because *e* is to *f* as (**g** to **h**, that is, as) **y** to **z**; and *f* is to *g* as (**k** to **l**, that is, as) **z** to *a*: therefore, *ex æquali*, *e* is to *g* as **y** to *a*. And by the hypothesis, **a** is to **b**, that is, **s** is to **t**, as *e* to *g*; wherefore **s** is to **t**, as **y** is to *a*; and, by inversion, **t** is to **s**, as *a* to **y**; and **s** is to **x**, as **y** to *d*; therefore, *ex æquali*, **t** is to **x**, as *a* to *d*. Also, because *h* is to *k* as (**c** to **d**, that is, as) **t** to **v**; and *k* is to *l*, as (**e** to **f**, that is, as) **v** to **x**; therefore, *ex æquali*, *h* is to *l*, as **t** to **x**. In like manner, it may be demonstrated that *m* is to *p*, as *a* to *d*: and it has been shewn that **t** is to **x** as *a* to *d*; therefore (v. 11) *h* is to *l*, as *m* to *p*. Q. E. D.

The propositions G and K are usually, for the sake of brevity, expressed in the same terms with propositions F and H: and therefore it was proper to shew the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers.

EXERCISES ON BOOK V.

THEOREMS.

1. If three magnitudes have the same ratio to each other, the sum of the first and third is greater than twice the second.

2. The differences of magnitudes that are continual proportionals are proportionals also.

3. If there be four proportional magnitudes of which the first is a multiple of the second, then the third is the same multiple of the fourth. (Prove also that if the second be a multiple of the first, the fourth is the same multiple of the third.)

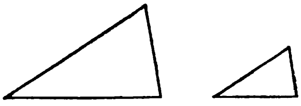
4. If the ratio between the first and second of four magnitudes to the second be greater than the ratio between the third and fourth to the fourth, then the first has a greater ratio to the second than the third has to the fourth.

5. If there be four proportional magnitudes, the ratio between the third and fourth is equal to, greater, or less, according as the first and second are equal to, greater, or less than each other.

BOOK VI.

DEFINITIONS.

I. SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.

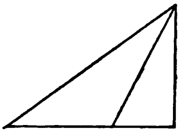


Definition I.

II. Reciprocal figures, viz. triangles and parallelograms, are such as have their sides about two of their angles proportionals in such manner that a side of the first figure is to a side of the other as the remaining side of this other is to the remaining side of the first.

III. A straight line is said to be cut in extreme and mean ratio when the whole is to the greater segment as the greater segment is to the less.

IV. The altitude of any figure is the straight line drawn from its vertex perpendicular to the base.



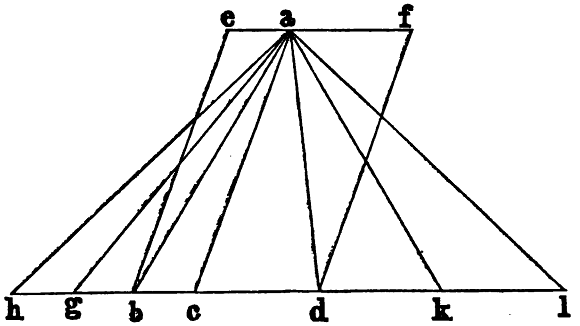
Definition IV.

PROPOSITION I.—THEOREM.

Triangles and parallelograms of the same altitude are one to another as their bases.

LET the triangles **a b c**, **a c d**, and the parallelograms **e c**, **c f**, have the same altitude viz. the perpendicular drawn from the point **a** to **b d** : then, as the base **b c** is to the base **c d**, so is the triangle **a b c** to the triangle **a c d**, and the parallelogram **e c** to the parallelogram **c f**.

Produce **b d** both ways to the points **h**, **l**, and take any number of straight lines **b g**, **g h**, each equal to the base **b c** ; and **d k**, **k l**, any number of them, each equal to the base **c d** ; and join **a g**, **a h**, **a k**, **a l** : then, because **c b**, **b g**, **g h** are all equal, the triangles **a h g**, **a g b**, **a b c**, are all equal (i. 38) : therefore, whatever multiple the base **h c** is of the base **b c**, the same multiple is the triangle **a h c** of the triangle **a b c**. For the same reason, whatever multiple the base **l c** is of the base **c d**, the same multiple is the triangle **a l c** of the triangle **a d c**: and if the base **h c** be equal to the base **c l**, the triangle **a h c** is also equal to the triangle **a l c** (i. 38): and if the base **h c** be greater than the base **c l**, likewise the triangle **a h c** is greater than the triangle **a l c**; and if less, less: therefore, since there are four magnitudes, viz. the two bases **b c**, **c d**, and the two triangles **a b c**, **a c d**; and of the base **b c**, and the triangle **a b c**, the first and third, any equimultiples whatever have been taken, viz. the base **h c** and triangle **a h c**; and of the base **c d** and triangle **a c d**, the second and fourth, have been taken any equimultiples whatever, viz. the base **c l** and triangle **a l c**; and that it has been shewn, that if the base **h c** be greater than the base **c l**, the triangle **a h c** is greater than the triangle **a l c**; and if equal, equal; and if less, less: therefore (v. *def.* 5), as the base **b c** is to the base **c d**, so is the triangle **a b c** to the triangle **a c d**.



And because the parallelogram **c e** is double of the triangle **a b c** (i. 41), and the parallelogram **c f** double of the triangle **a c d**, and that magnitudes have the same ratio which their equimultiples have (v. 15); as the triangle **a b c** is to the triangle **a c d**, so is the parallelogram **e c** to the parallelogram **c f**; and because it has been shewn, that as the base **b c** is to the base **c d**, so is the triangle **a b c** to the triangle **a c d**; and as the triangle **a b c** is to the triangle **a c d**, so is the parallelogram **e c** to the parallelogram **c f**; therefore, as the base **b c** is to the base **c d**, so is (v. 11) the parallelogram **e c** to the parallelogram **c f**. Wherefore, triangles, &c. Q. E. D.

COR. From this it is plain, that triangles and parallelograms that have equal altitudes are one to another as their bases.

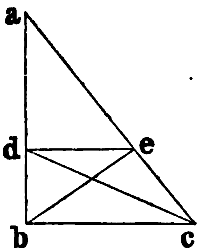
Let the figures be placed so as to have their bases in the same straight line; and having drawn perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are (i. 33), because the perpendiculars are both equal and parallel to one another. Then, if the same construction be made as in the proposition, the demonstration will be the same.

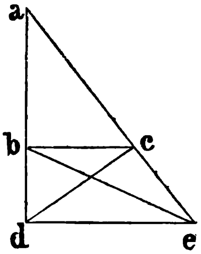
PROPOSITION II.—THEOREM.

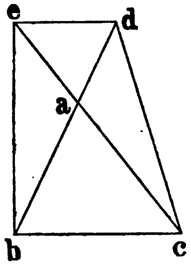
If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or these produced, proportionally ; and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

LET d e be drawn parallel to b c, one of the sides of the triangle a b c : b d is to d a, as c e to e a.

Join b e, c d ; then the triangle b d e is equal to the triangle c d e (i. 37), because they are on the same base d e, and between the same parallels d e, b c. But a d e is another triangle, and equal magnitudes have to the same the same ratio (v. 7) ; therefore, as the triangle b d e to the triangle a d e, so is the triangle c d e to the triangle a d e, but as the triangle b d e to the triangle a d e, so is (vi. 1) b d to d a, because having the same altitude, viz. the perpendicular drawn from the point e to a b, they are to one another as their bases ; and for the same reason, as the triangle c d e to the triangle a d e, so is c e to e a. Therefore, as b d to d a, so is c e to e a (v. 11).







Next, let the sides a b, a c, of the triangle a b c, or these produced, be cut proportionally in the points d, e, that is, so that b d be to d a as c e to e a, and join d e ; d e is parallel to b c.

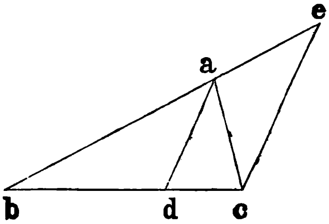
The same construction being made, because as b d to d a, so is c e to e a ; and as b d to d a, so is the triangle b d e to the triangle a d e (vi. 1) ; and as c e to e a, so is the triangle c d e to the triangle a d e ; therefore the triangle b d e is to the triangle a d e, as the triangle c d e to the triangle a d e ; that is, the triangles b d e, c d e have the same ratio to the triangle a d e ; and therefore (v. 9) the triangle b d e is equal to the triangle c d e ; and they are on the same base d e : but equal triangles on the same base are between the same parallels (i. 39) ; therefore d e is parallel to b c. Wherefore, if a straight line, &c. Q. E. D.

PROPOSITION III.—THEOREM.

If the angle of a triangle be divided into two equal angles by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another ; and if the segments of the base have the same ratio which the other sides of the triangle *have to one another, the straight line drawn from the vertex to the point of section divides the vertical angle into two equal angles.*

LET the angle **bac** of any triangle **abc** be divided into two equal angles by the straight line **ad** : **bd** is to **dc** as **ba** to **ac**.

Through the point **c** draw **ce** parallel (i. 31) to **da**, and let **ba** produced meet **ce** in **e**; because the straight line **ac** meets the parallels **ad**, **ec**, the angle **ace** is equal to the alternate angle **cad** (i. 29) : but **cad**, by the hypothesis, is equal to the angle **bad**; wherefore **bad** is equal to the angle **ace**. Again, because the straight line **bae** meets the parallels **ad**, **ec**, the outward angle **bad** is equal to the inward and opposite angle **aec** : but the angle **ace** has been proved equal to the angle **bad**; therefore also **ace** is equal to the angle **aec**, and consequently the side **ae** is equal to the side **ac** (i. 6) : and because **ad** is drawn parallel to one of the sides of the triangle **bce**, viz. to **ec**, **bd** is to **dc**, as **ba** to **ae** (vi. 2), but **ae** is equal to **ac**; therefore as **bd** to **dc**, so is **ba** to **ac** (v. 7).



Let now **bd** be to **dc**, as **ba** to **ac**, and join **ad**; the angle **bac** is divided into two equal angles by the straight line **ad**.

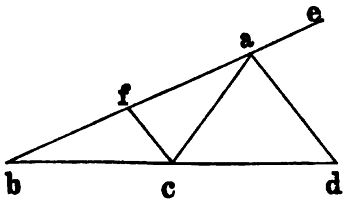
The same construction being made; because, as **bd** to **dc**, so is **ba** to **ac**; and as **bd** to **cd** so is **ba** to **ae** (vi. 2), because **ad** is parallel to **ec**; therefore **ba** is to **ac**, as **ba** to **ae** (v. 11); consequently **ac** is equal to **ae** (v. 9), and the angle **aec** is therefore equal to the angle **ace** (i. 5) : but the angle **aec** is equal to the outward and opposite angle **bad**; and the angle **ace** is equal to the alternate angle **cad** (i. 29) : wherefore also the angle **bad** is equal to the angle **cad**; therefore the angle **bac** is cut into two equal angles by the straight line **ad**. Therefore, if the angle, &c. Q. E. D.

PROPOSITION A.—THEOREM.

If the outward angle of a triangle made by producing one of its sides be divided into two equal angles by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to one another; and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

LET the outward angle **cae** of any triangle **abc** be divided into two qual angles by the straight line **ad** which meets the base produced in : **bd** is to **dc**, as **ba** to **ac**.

Through **c** draw **cf** parallel to **ad** (i. 31); and because the straight line **ac** meets the parallels **ad**, **cf**, the angle **acf** is equal to the alternate angle **cad** (i. 29): but **cad** is equal to the angle **dae** (*hyp.*); therefore also **dae** is equal to the angle **acf**. Again, because the straight line **fae** meets the parallels **ad**, **fc**, the outward angle **dae** is equal to the inward and opposite angle **cfa**: but the angle **acf** has been proved equal to the angle **dae**; therefore also the angle **acf** is equal to the angle **cfa**, and consequently the side **af** is equal to the side **ac** (i. 6): and because **ad** is parallel to **fc**, a side of the triangle **bcf**, **bd** is to **dc**, as **ba** to **af** (vi. 2); but **af** is equal to **ac**; as therefore **bd** is to **dc**, so is **ba** to **ac**.



Let now **bd** be to **dc** as **ba** to **ac**, and join **ad**; the angle **cad** is equal to the angle **dae**.

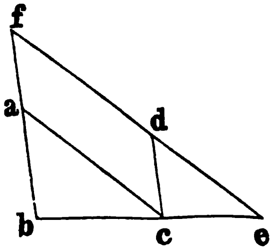
The same construction being made, because **bd** is to **dc** as **ba** to **ac**; and that **bd** is also to **dc**, as **ba** to **af** (vi. 2); therefore **ba** is to **ac** as **ba** to **af** (v. 11); wherefore **ac** is equal to **af** (v. 9), and the angle **afc** equal (i. 5) to the angle **acf**: but the angle **afc** is equal to the outward angle **ead**, and the angle **acf** to the alternate angle **cad**; therefore also **ead** is equal to the angle **cad**. Wherefore, if the outward, &c. Q. E. D.

PROPOSITION IV.—THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

LET **abc**, **dce** be equiangular triangles, having the angle **abc** equal to the angle **dce**, and the angle **acb** to the angle **dec**, and consequently (i. 32) the angle **bac** equal to the angle **cde**. The sides about the equal angles of the triangles **abc**, **dce** are proportionals; and those are the homologous sides which are opposite to the equal angles.

Let the triangle **dce** be placed so that its side **ce** may be contiguous to **bc**, and in the same straight line with it: and because the angles **abc**, **acb**, are together less than two right angles (i. 17), **abc** and **dec**, which is equal to **acb**, are also less than two right angles; wherefore **ba**, **ed** produced shall meet (i. *ax.* 12); let them be produced and meet in the point **f**; and because the angle **abc** is equal to the angle **dce**, **bf** is parallel to **cd** (i. 28). Again, because the angle **acb** is equal to the angle **dec**, **ac** is parallel to **fe** (i. 28): therefore **facd** is a parallelogram, and consequently **af** is equal to **cd**, and **ac** to **fd** (i. 34): and because **ac** is parallel to **fe**, one of the sides of the triangle **fbe**, **ba** is to **af** as **bc** to **ce** (vi. 2): but **af** is equal to **cd**; therefore (v. 7), as **ba** to **cd**, so is **bc** to **ce**; and alternately, as **ab** to **bc**, so is **dc** to **ce**.



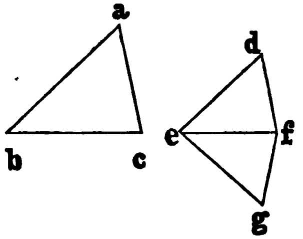
Again, because **cd** is parallel to **bf**, as **bc** to **ce**, so is **fd** to **de** (vi. 2); but **fd** is equal to **ac**; therefore, as **bc** to **ce**, so is **ac** to **de**: and alternately, as **bc** to **ca**, so is **ce** to **ed**: therefore, because it has been proved that **ab** is to **bc**, as **dc** to **ce**, and as **bc** to **ca**, so is **ce** to **ed**, *ex æquali* (v. 22), **ba** is to **ac**, as **cd** to **de**. Therefore the sides, &c. Q. E. D.

PROPOSITION V.—THEOREM.

If the sides of two triangles about each of their angles be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

LET the triangles **abc**, **def** have their sides proportionals, so that **ab** is to **bc** as **de** to **ef**; and **bc** to **ca** as **ef** to **fd**; and consequently, *ex æquali*, **ba** to **ac** as **ed** to **df**; the triangle **abc** is equiangular to the triangle **def**, and their equal angles are opposite to the homologous sides, viz. the angle **abc** equal to the angle **def**, and **bca** to **efd**, and also **bac** to **edf**.

At the points **e**, **f**, in the straight line **ef**, make (i. 23) the angle **feg** equal to the angle **abc**, and the angle **efg** equal to **bca**; wherefore the remaining angle **bac** is equal to the remaining angle **egf** (i. 32), and the triangle **abc** is therefore equiangular to the triangle **gef**; and consequently they have their sides opposite to the equal angles proportionals (vi. 4). Wherefore as **ab** to **bc**, so is **ge** to **ef**; but as **ab** to **bc**, so is **de** to **ef**; therefore as **de** to **ef**, so is **ge** to **ef** (v. 11): therefore **de** and **ge** have the same ratio to **ef**, and consequently are equal (v. 9): for the same reason, **df** is equal to **fg**: and because, in the triangles **def**, **gef**, **de** is equal to **eg**, and **ef** common, the two sides **de**, **ef**, are equal to the two **ge**, **ef**, and the base **df** is equal to the base **gf**; therefore the angle **def** is equal (i. 8) to the angle **gef**, and the other angles to the other angles which are subtended by the equal sides (i. 4). Wherefore the angle **dfe** is equal to the angle **gfe**, and **edf** to **egf**: and because the angle **def** is equal to the angle **gef**, and **gef** to the angle **abc**; therefore the angle **abc** is equal to the angle **def**. For the same reason, the angle **acb** is equal to the angle **dfe**, and the angle at **a** to the angle at **d**. Therefore the triangle **abc** is equiangular to the triangle **def**. Wherefore, if the sides, &c. Q. E. D.

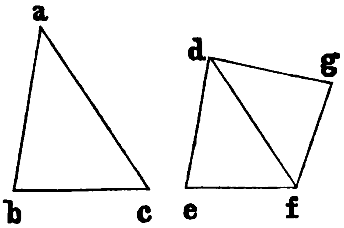


PROPOSITION VI.—THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be *equiangular, and shall have those angles equal which are opposite to the homologous sides.*

Let the triangles **a b c**, **d e f** have the angle **b a c** in the one equal to the angle **e d f** in the other, and the sides about those angles proportionals; that is, **b a** to **a c**, as **e d** to **d f**; the triangles **a b c**, **d e f** are equiangular, and have the angle **a b c** equal to the angle **d e f**, and **a c b** to **d f e**.

At the points **d**, **f**, in the straight line **d f**, make (i. 23) the angle **f d g** equal to either of the angles **b a c**, **e d f**; and the angle **d f g** equal to the angle **a c b**: wherefore the remaining angle at **b** is equal to the remaining one at **g** (i. 32), and consequently the triangle **a b c** is equiangular to the triangle **d g f**; and therefore as **b a** to **a c**, so is (vi. 4) **g d** to **d f**; but by the hypothesis, as **b a** to **a c**, so is **e d** to **d f**; as therefore **e d** to **d f**, so is (v. 11) **g d** to **d f**; wherefore **e d** is equal (v. 9) to **d g**; and **d f** is common to the two triangles **e d f**, **g d f**: therefore the two sides **e d**, **d f** are equal to the two sides **g d**, **d f**; and the angle **e d f** is equal to the angle **g d f**; wherefore the base **e f** is equal to the base **f g** (i. 4) and the triangle **e d f** to the triangle **g d f**, and the remaining angles to the remaining angles, each to each, which are subtended by the equal sides; therefore the angle **d f g** is equal to the angle **d f e**, and the angle at **g** to the angle at **e**: but the angle **d f g** is equal to the angle **a c b**, therefore the angle **a c b** is equal to the angle **d f e**: and the angle **b a c** is equal to the angle **e d f** (*hyp.*); wherefore also the remaining angle at **b** is equal to the remaining angle at **e**. Therefore the triangle **a b c** is equiangular to the triangle **d e f**. Wherefore, if two triangles, &c. Q. E. D.

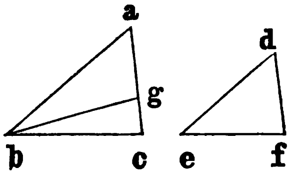


Proposition VII.—Theorem.

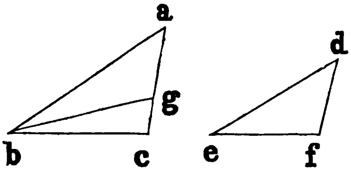
If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Let the two triangles **a b c**, **d e f** have one angle in the one equal to one angle in the other, viz. the angle **b a c** to the angle **e d f**, and the sides about two other angles **a b c**, **d e f** proportionals, so that **a b** is to **b c** as **d e** to **e f**; and in the first case, let each of the remaining angles at **c**, **f** be less than a right angle. The triangle **a b c** is equiangular to the triangle **d e f**, viz. the angle **a b c** is equal to the angle **d e f**, and the remaining angle at **c** to the remaining angle at **f**.

For if the angles **a b c**, **d e f** be not equal, one of them is greater than the other: let **a b c** be the greater, and at the point **b**, in the straight line **a b**, make the angle **a b g** equal to the angle (i. 23) **d e f**: and because the angle at **a** is equal to the angle at **d**, and the angle **a b g** to the angle **d e f**; the remaining angle **a g b** is equal (i. 32) to the remaining angle **dfe**: therefore the triangle **abg** is equiangular to the triangle **def**; wherefore (vi. 4) as **ab** is to **bg**, so is **de** to **ef**, but as **de** to **ef**, so, by hypothesis, is **ab** to **bc**; therefore as **ab** to **bc**, so is **ab** to **bg** (v. 11): and because **ab** has the same ratio to each of the lines **bc**, **bg**; **bc** is equal (v. 9) to **bg**, and therefore the angle **bgc** is equal to the angle **bcg** (i. 5): but the angle **bcg** is, by hypothesis, less than a right angle; therefore also the angle **bgc** is less than a right angle, and the adjacent angle **agb** must be greater than a right angle (i. 13). But it was proved that the angle **agb** is equal to the angle at **f**; therefore the angle at **f** is greater than a right angle: but, by the hypothesis, it is less than a right angle; which is absurd. Therefore the angles **abc**, **def** are not unequal, that is, they are equal: and the angle at **a** is equal to the angle at **d**; wherefore the remaining angle at **c** is equal to the remaining angle at **f**: therefore the triangle **abc** is equiangular to the triangle **def**.



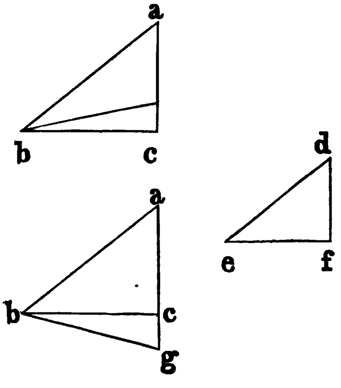
Next, let each of the angles at **c**, **f** be not less than a right angle: the triangle **abc** is also in this case equiangular to the triangle **def**.



The same construction being made, it may be proved in like manner that **bc** is equal to **bg**, and the angle at **c** equal to the angle **bgc**: but the angle at **c** is not less than a right angle; therefore the angle **bgc** is not less than a right angle: wherefore two angles of the triangle **bgc** are together not less than two right angles, which is impossible (i. 17); and therefore the triangle **abc** may be proved to be equiangular to the triangle **def**, as in the first case.

Lastly, let one of the angles at **c**, **f**, viz. the angle at **c**, be a right angle; in this case likewise the triangle **abc** is equiangular to the triangle **def**.

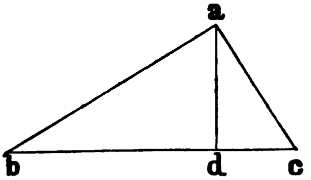
For if they be not equiangular, make, at the point **b** of the straight line **ab**, the angle **abg** equal to the angle **def**; then it may be proved, as in the first case, that **bg** is equal to **bc**: but the angle **bcg** is a right angle, therefore (i. 5) the angle **bgc** is also a right angle; whence two of the angles of the triangle **bgc** are together not less than two right angles, which is impossible (i. 17): therefore the triangle **abc** is equiangular to the triangle **def**. Wherefore, if two triangles, &c. Q. E. D.



PROPOSITION VIII.—THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.

LET **a b c** be a right-angled triangle, having the right angle **b a c**; and from the point **a** let **a d** be drawn perpendicular to the base **b c**: the triangles **a b d**, **a d c** are similar to the whole triangle **a b c**, and to one another.



Because the angle **b a c** is equal to the angle **a d b**, each of them being a right angle, and that the angle at **b** is common to the two triangles **a b c**, **a b d**; the remaining angle **a c b** is equal to the remaining angle **b a d** (i. 32): therefore the triangle **a b c** is equiangular to the triangle **a b d**, and the sides about their equal angles are proportionals (vi. 4); wherefore the triangles are similar (vi. *def.* 1): in the like manner it may be demonstrated, that the triangle **a d c** is equiangular and similar to the triangle **a b c**: and the triangles **a b d**, **a c d**, being both equiangular and similar to **a b c**, are equiangular and similar to each other. Therefore, in a right-angled triangle, &c. Q. E. D.

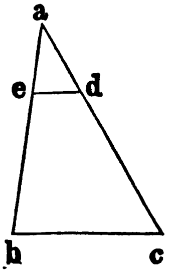
COR. From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base: and also, that each of the sides is a mean proportional between the base, and its segment adjacent to that side: because in the triangles **b d a**, **a d c**, **b d** is to **d a**, as **d a** to **d c** (vi. 4); and in the triangles **a b c**, **d b a**, **b c** is to **b a**, as **b a** to **b d** (vi. 4); and in the triangles **a b c**, **a c d**, **b c** is to **c a**, as **c a** to **c d** (vi. 4).

PROPOSITION IX.—PROBLEM.

From a given straight line to cut off any part required.

LET **a b** be the given straight line; it is required to cut off any part from it.

From the point **a** draw a straight line **a c**, making any angle with **a b**; and in **a c** take any point **d**, and take **a c** the same multiple of **a d**, that **a b** is of the part which is to be cut off from it: join **b c**, and draw **d e** parallel to it: then **a e** is the part required to be cut off.



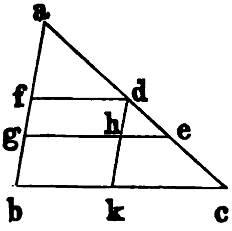
Because **e d** is parallel to one of the sides of the triangle **a b c**, viz. to **b c**, as **c d** is to **d a**, so is **b e** to **e a** (vi. 2); and by composition (v. 18), **c a** is to **a d**, as **b a** to **a e**: but **c a** is a multiple of **a d**; therefore (v. D) **b a** is the same multiple of **a e**: whatever part therefore **a d** is of **a c**, **a e** is the same part of **a b**: wherefore, from the straight line **a b** the part required is cut off. Which was to be done.

PROPOSITION X.—PROBLEM.

To divide a given straight line similarly to a given divided straight line, that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

LET **ab** be the straight line given to be divided, and **ac** the divided line: it is required to divide **ab** similarly to **ac**.

Let **ac** be divided in the points **d**, **e**; and let **ab**, **ac** be placed so as to contain any angle, and join **bc**, and through the points **d**, **e**, draw (i. 31) **df**, **eg** parallels to it; and through **d** draw **dhk** parallel to **ab**: therefore each of the figures **fh**, **hb**, is a parallelogram: wherefore **dh** is equal (i. 34) to **fg**, and **hk** to **gb**: and because **he** is parallel to **kc**, one of the sides of the triangle **dkc**, as **ce** to **ed**, so is (vi. 2) **kh** to **hd**: but **kh** is equal to **bg**, and **hd** to **gf**; therefore, as **ce** to **ed**, so is **bg** to **gf**. Again, because **fd** is parallel to **eg**, one of the sides of the triangle **age**, as **ed** to **da**, so is **gf** to **fa**: but it has been proved that **ce** is to **ed** as **bg** to **gf**; and as **ed** to **da**, so is **gf** to **fa**: therefore the given straight line **ab** is divided similarly to **ac**. Which was to be done.



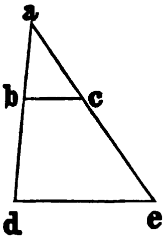
PROPOSITION XI.—PROBLEM.

To find a third proportional to two given straight lines.

LET **ab**, **ac** be the two given straight lines, and let them be placed so as to contain any angle; it is required to find a third proportional to **ab**, **ac**.

Produce **ab**, **ac**, to the points **d**, **e**; and make **bd** equal to **ac**; and having joined **bc**, through **d** draw **de** parallel to it (i. 31).

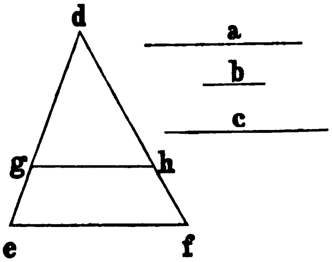
Because **bc** is parallel to **de**, a side of the triangle **ade**, **ab** is (vi. 2) to **bd**, as **ac** to **ce**: but **bd** is equal to **ac**; as therefore **ab** to **ac**, so is **ac** to **ce**. Wherefore, to the two given straight lines **ab**, **ac**, a third proportional **ce** is found. Which was to be done.



PROPOSITION XII.—PROBLEM.

To find a fourth proportional to three given straight lines.

LET **a**, **b**, **c** be the three given straight lines; it is required to find a fourth proportional to **a**, **b**, **c**.

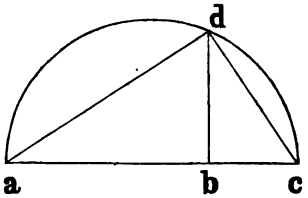


Take two straight lines **de**, **df**, containing any angle **edf**; and upon these make **dg** equal to **a**, **ge** equal to **b**, and **dh** equal to **c**; and having joined **gh**, draw **ef** parallel (i. 31) to it through the point **e**: and because **gh** is parallel to **ef**, one of the sides of the triangle **def**, **dg** is to **ge** as **dh** to **hf** (vi. 2); but **dg** is equal to **a**, **ge** to **b**, and **dh** to **c**; therefore, as **a** is to **b**, so is **c** to **hf**. Wherefore to the three given straight lines **a**, **b**, **c**, a fourth proportional **hf** is found. Which was to be done.

PROPOSITION XIII.—PROBLEM.

To find a mean proportional between two given straight lines.

LET **ab**, **bc** be the two given straight lines: it is required to find a mean proportional between them.



Place **ab**, **bc** in a straight line, and upon **ac** describe the semicircle **adc**, and from the point **b** draw (i. 11) **bd** at right angles to **ac**, and join **ad**, **dc**.

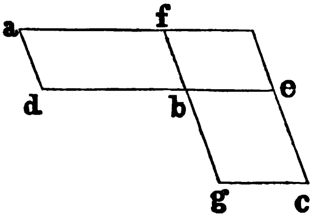
Because the angle **adc** in a simicircle is a right angle (iii. 31), and because in the right-angled triangle **adc**, **db** is drawn from the right angle perpendicular to the base, **db** is a mean proportional between **ab**, **bc** the segments of the base (vi. 3. *cor.*): therefore between the two given straight lines **ab**, **bc**, a mean proportional **db** is found. Which was to done.

PROPOSITION XIV.—THEOREM.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

LET **ab**, **bc** be equal parallelograms, which have the angles at **b** equal, and let the sides **db**, **be** be placed in the same straight line; wherefore also **fb**, **bg** are in one straight line (i. 14). The sides of the parallelograms **a b**, **b c** about the equal angles, are reciprocally proportional; that is, **d b** is to **b e** as **g b** to **b f**.

Complete the parallelogram **f e**; and because the parallelogram **a b** is equal to **b c**, and that **f e** is another parallelogram, **a b** is to **f e** as **b c** to **f e** (v. 7): but as **a b** to **f e**, so is the base **d b** to **b e** (vi. 1): and as **b c** to **f e**, so is the base of **g b** to **b f**; therefore, as **d b** to **b e**, so is **g b** to **b f** (v. 11). Wherefore, the sides of the parallelograms **a b**, **b c** about their equal angles are reciprocally proportional.



But, let the sides about the equal angles be reciprocally proportional, viz. as **d b** to **b e**, so **g b** to **b f**; the parallelogram **a b** is equal to the parallelogram **b c**.

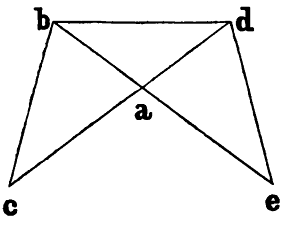
Because, as **d b** to **b e**, so is **g b** to **b f**; and as **d b** to **b e**, so is the parallelogram **a b** to the parallelogram **f e**; and as **g b** to **b f**, so is the parallelogram **b c** to the parallelogram **f e**; therefore as **a b** to **f e**, so is **b c** to **f e** (v. 11): wherefore the parallelogram **a b** is equal (v. 9) to the parallelogram **b c**. Therefore equal parallelograms, &c. Q. E. D.

Proposition XV.—Theorem.

Equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Let **a b c**, **a d e** be equal triangles, which have the angle **b a c** equal to the angle **d a e**; the sides about the equal angles of the triangles are reciprocally proportional; that is, **c a** is to **a d**, as **e a** to **a b**.

Let the triangles be placed so that their sides **c a**, **a d** be in one straight line; wherefore also **e a** and **a b** are in one straight line (i. 14); and join **b d**. Because the triangle **a b c** is equal to the triangle **a d e**, and that **a b d** is another triangle; therefore as the triangle **c a b**, is to the triangle **b a d**, so is the triangle **e a d** to triangle **d a b** (v. 7): but as triangle **c a b** to triangle **b a d**, so is the base **c a** to **a d** (vi. 1); and as triangle **e a d** to triangle **d a b**, so is the base **e a** to **a b** (vi. 1): as therefore **c a** to **a d**, so is **e a** to **a b** (v. 11); wherefore the sides of the triangles **a b c**, **a d e** about the equal angles are reciprocally proportional.



But let the sides of the triangles **a b c**, **a d e** about the equal angles be reciprocally proportional, viz. **c a** to **a d**, as **e a** to **a b**; the triangle **a b c** is equal to the triangle **a d e**.

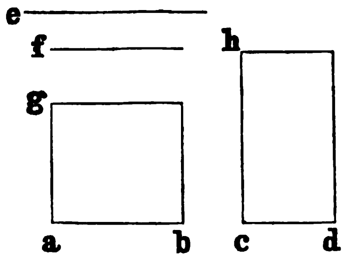
Having joined **b d** as before; because, as **c a** to **a d**, so is **e a** to **a b**; and as **c a** to **a d**, so is triangle **a b c** to triangle **b a d** (vi. 1); and as **e a** to **a b**, so is triangle **e a d** to triangle **b a d** (vi. 1); therefore (v. 11) as triangle **b a c** to triangle **b a d**, so is triangle **e a d** to triangle **b a d**: that is, the triangles **b a c**, **e a d** have the same ratio to the triangle **b a d**: wherefore the triangle **a b c** is equal (v. 9) to the triangle **a d e**. Therefore equal triangles, &c. Q. E. D.

Proposition XVI.—Theorem.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Let the four straight lines **a b**, **c d**, **e**, **f**, be proportionals, viz. as **a b** to **c d**, so **e** to **f**; the rectangle contained by **a b**, **f** is equal to the rectangle contained by **c d**, **e**.

From the points **a**, **c**, draw (i. 11) **a g**, **c h** at right angles to **a b**, **c d**; and make **a g** equal to **f**, and **c h** equal to **e**, and complete the parellelograms **b g**, **d h**; because, as **a b** to **c d**, so is **e** to **f**; and that **e** is equal to **c h**, and **f** to **a g**; **a b** is (v. 7) to **c d** as **c h** to **a g**. Therefore the sides of the parallelograms **b g**, **d h** about the equal angles are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another (vi. 14); therefore the parallelogram **b g** is equal to the parallelogram **d h**: and the parallelogram **b g** is contained by the straight lines **a b**, **f**; because **a g** is equal to **f**; and the parallelogram **d h** is contained by **c d** and **e**; because **c h** is equal to **e**: therefore the rectangle contained by the straight lines **a b**, **f** is equal to that which is contained by **c d** and **e**.



And if the rectangle contained by the straight lines **a b**, **f** be equal to that which is contained by **c d**, **e**; these four lines are proportionals, viz. **a b** is to **c d** as **e** to **f**.

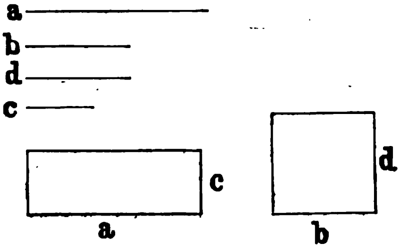
The same construction being made, because the rectangle contained by the straight lines **a b**, **f** is equal to that which is contained by **c d**, **e**, and that the rectangle **b g** is contained by **a b**, **f**, because **a g** is equal to **f**; and the rectangle **d h** by **c d**, **e**, because **c h** is equal to **e**; therefore the parallelogram **b g** is equal to the parallelogram **d h**; and they are equiangular: but the sides about the equal angles of equal parallelograms are reciprocally proportional (vi. 14): wherefore, as **a b** to **c d**, so is **c h** to **a g**; and **c h** is equal to **e**, and **a g** to **f**: as therefore **a b** is to **c d**, so is **e** to **f**. Wherefore, if four, &c. Q. E. D.

PROPOSITION XVII.—THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean; and if the rectangle contained by the extremes be equal to the square of the mean, the three straight lines are proportionals.

LET the three straight lines a, b, c be proportionals, viz. as a to b so b to c; the rectangle contained by a, c is equal to the square of b.

Take d equal to b; and because as a to b so b to c, and that b is equal to d: a is (v. 7) to b as d to c: but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (vi. 16): therefore the rectangle contained by a, c is equal to that contained by b, d: but the rectangle contained by b, d is the square of b; because b is equal to d: therefore the rectangle contained by a, c is equal to the square of b.



And if the rectangle contained by a, c be equal to the square of b; a is to b as b is to c.

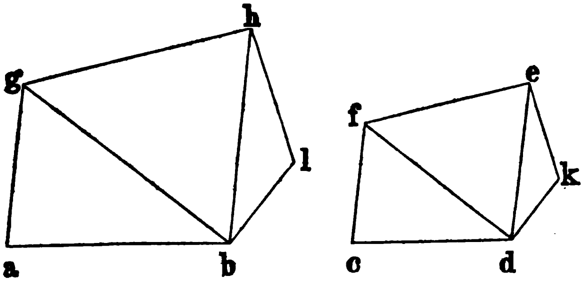
The same construction being made, because the rectangle contained by a, c is equal to the square of b, and the square of b is equal to the rectangle contained by b, d, because b is equal to d; therefore the rectangle contained by a, c is equal to that contained by b, d; but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals (vi. 16): therefore a is to b as d to c; but b is equal to d; wherefore, as a to b, so b to c: therefore, if three straight lines, &c. Q. E. D.

PROPOSITION XVIII.—THEOREM.

Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

LET a b be the given straight line, and c d e f the given rectilineal figure of four sides; it is required upon the given straight line a b to describe a rectilineal figure similar, and similarly situated, to c d e f.

Join d f, and at the points a, b in the straight line a b make (i. 23) the angle b a g equal to the angle at c, and the angle a b g equal to the angle c d f; therefore the remaining angle c f d is equal to the remaining angle a g b (i. 32). Wherefore the triangle f c d is equiangular to the triangle g a b: again, at the points g, b in the straight line g b, make (i. 23) the angle b g h equal to the angle d f e, and the angle g b h equal to f d e; therefore the remaining angle f e d is equal to the remaining angle g h b, and the triangle f d e equiangular to the triangle g b h: then, because the angle **agb** is equal to the angle **cfd**, and **bgh** to **dfe**, the whole angle **agh** is equal to the whole **cfe**: for the same reason, the angle **abh** is equal to the angle **cde**; also the angle at **a** is equal to the angle at **c**, and the angle **ghb** to **fed**: therefore the rectilineal figure **abhg** is equiangular to **cdef**: but likewise these figures have their sides about the equal angles proportionals; because the triangles **gab**, **fcd** being equiangular, **ba** is (vi. 4) to **ag** as **dc** to **cf**; and because **ag** is to **gb** as **cf** to **fd**; and as **gb** to **gh**, so, by reason of the equiangular triangles **bgh**, **dfe**, is **fd** to **fe**; therefore, *ex æquali* (v. 22), **ag** is to **gh** as **cf** to **fe**: in the same manner it may be proved that **ab** is to **bh** as **cd** to **de**: and **gh** is to **hb** as **fe** to **ed** (vi. 4). Wherefore, because the rectilineal figures **abhg**, **cdef** are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (vi. *def.* 1).



Next, let it be required to describe upon a given straight line **ab**, a rectilineal figure similar, and similarly situated, to the rectilineal figure **cdkef**.

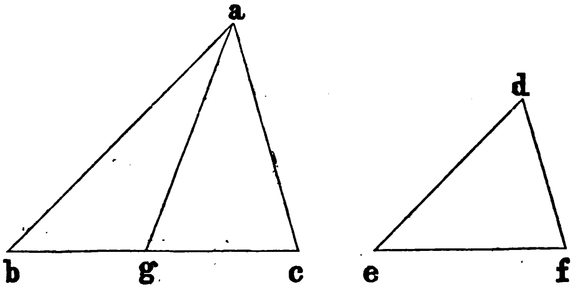
Join **de**, and upon the given straight line **ab** describe the rectilineal figure **abhg** similar, and similarly situated, to the quadrilateral figure **cdef**, by the former case; and at the points **b**, **h**, in the straight line **bh**, make the angle **hbl** equal to the angle **edk**, and the angle **bhl** equal to the angle **dek**; therefore the remaining angle at **k** is equal to the remaining angle at **l**: and because the figures **abhg**, **cdef** are similar, the angle **ghb** is equal to the angle **fed**, and **bhl** is equal to **dek**; wherefore the whole angle **ghl** is equal to the whole angle **fek**: for the same reason the angle **abl** is equal to the angle **cdk**: therefore the five-sided figures **aghlb**, **cfekd** are equiangular; and because the figures **aghb**, **cfed** are similar, **gh** is to **hb** as **fe** to **ed**; and as **hb** to **hl**, so is **ed** to **ek** (vi. 4); therefore, *ex æquali* (v. 22) **gh** is to **hl**, as **fe** to **ek**: for the same reason, **ab** is to **bl** as **cd** to **dk**: and **bl** is to **lh** as (vi. 4) **dk** to **ke**, because the triangles **blh**, **dke** are equiangular: therefore because the five-sided figures **aghlb**, **cfekd** are equiangular, and have their sides about the equal angles proportionals, they are similar to one another; and in the same manner a rectilineal figure of six or more sides may be described upon a given straight line similar to a given rectilineal figure, and so on. Which was to be done.

Proposition XIX.—Theorem.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let **abc**, **def** be similar triangles, having the angle **b** equal to the angle **e**, and let **ab** be to **bc**, as **de** to **ef**, so that the side **bc** is homologous to **ef** (v. *def.* 12): the triangle **abc** has to the triangle **def**, the duplicate ratio of that which **bc** has to **ef**.

Take **bg** a third proportional to **bc**, **ef** (vi. 11) so that **bc** is to **ef** as **ef** to **bg**, and join **ga**: then, because, as **ab** to **bc**, so **de** to **ef**; alternately (v. 16) **ab** is to **de** as **bc** to **ef**: but as **bc** to **ef**, so is **ef** to **bg**; therefore (v. 11) as **ab** to **de**, so is **ef** to **bg**. Wherefore the sides of the triangles **abg**, **def**, which are about the equal angles, are reciprocally proportional: but triangles which have the sides about two equal angles reciprocally proportional are equal to one another (vi. 15): therefore the triangle **abg** is equal to the triangle **def**: and because as **bc** is to **ef**, so is **ef** to **bg**; and that if three straight lines be proportionals, the first is said (v. *def.* 10) to have to the third the duplicate ratio of that which it has to the second; **bc** therefore has to **bg** the duplicate ratio of that which **bc** has to **ef**: but as **bc** to **bg**, so is (vi. 1) the triangle **abc** to the triangle **abg**. Therefore the triangle **abc** has to the triangle **abg** the duplicate ratio of that which **bc** has to **ef**: but the triangle **abg** is equal to the triangle **def**: wherefore also the triangle **abc** has to the triangle **def** the duplicate ratio of that which **bc** has to **ef**. Therefore similar triangles, &c. Q. E. D.



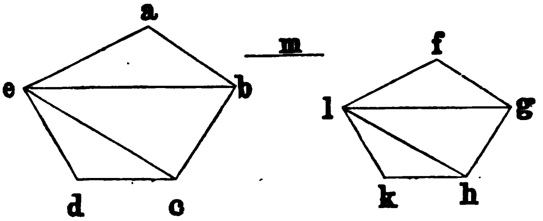
Cor. From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

Proposition XX.—Theorem.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Let **abcde**, **fghkl** be similar polygons, and let **ab** be the homologous side to **fg**: the polygons **abcde, fghkl** may be divided into the same number of similar triangles, whereof each to each has the same ratio which the polygons have; and the polygon **abcde** has to the polygon **fghkl** the duplicate ratio of that which the side **ab** has to the side **fg**.

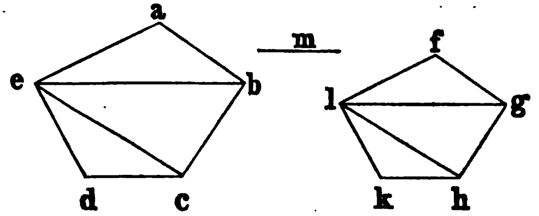
Join **be, ec, gl, lh**: and because the polygon **abcde** is similar to the polygon **fghkl**, the angle **bae** is equal to the angle **gfl** (vi. *def.* 1). and **ba** is to **ae**, as **gf** to **fl** (vi. *def.* 1): wherefore, because the triangles **abe, fgl** have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle **abe** is equiangular (vi. 6), and therefore similar, to the triangle **fgl** (vi. 4); wherefore the angle **abe** is equal to the angle **fgl**: and, because the polygons are similar, the whole angle **abc** is equal (vi. *def.* 1) to the whole angle **fgh**; therefore the remaining angle **ebc** is equal to the remaining angle **lgh**: and because the triangles **abe, fgl** are similar, **eb** is to **ba**, as **lg** to **gf** (vi. *def.* 1); and also, because the polygons are similar, **ab** is to **bc**, as **fg** to **gh** (vi. *def.* 1); therefore, *ex æquali* (v. 22) **eb** is to **bc**, as **lg** to **gh**: that is, the sides about the equal angles **ebc, lgh** are proportionals; therefore (v. 22) the triangle **ebc** is equiangular to the triangle **lgh**, and similar to it (vi. 4).



For the same reason, the triangle **ecd** likewise is similar to the triangle **lhk**: therefore the similar polygons **abcde, fghkl** are divided into the same number of similar triangles.

Also these triangles have each to each the same ratio which the polygons have to one another, the antecedents being **abe, ebc, ecd**, and the consequents **fgl, lgh, lhk**: and the polygon **abcde** has to the polygon **fghkl** the duplicate ratio of that which the side **ab** has to the homologous side **fg**.

Because the triangle **abe** is similar to the triangle **fgl**, **abe** has to **fgl**, the duplicate ratio (vi. 19) of that which the side **be** has to the side **gl**: for the same reason, the triangle **bec** has to **glh** the duplicate ratio of that which **be** has to **gl**: therefore, as the triangle **abe** to the triangle **fgl**, so (v. 11) is the triangle **bec** to the triangle **glh**. Again, because the triangle **ebc** is similar to the triangle **lgh**, **ebc** has to **lgh** the duplicate ratio of that which the side **ec** has to the side **lh**: for the same reason, the triangle **ecd** has to the triangle **lhk** the duplicate ratio of that which **ec** has to **lh**: as therefore the triangle **ebc** to the triangle **lgh**, so is (v. 11) the triangle **ecd** to the triangle **lhk**: but it has been proved that the triangle **ebc** is likewise to the triangle **lgh**, as the triangle **abe** to the triangle **fgl**. Therefore, as the triangle **abe** is to the triangle **fgl**, so is triangle **ebc** to triangle **lgh**, and triangle **ecd** to triangle **l h k**: and therefore, as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (v. 12). Wherefore, as the triangle **a b e** to the triangle **f g l**, so is the polygon



a b c d e to the polygon **f g h k l**: but the triangle **a b e** has to the triangle **f g l**, the duplicate ratio of that which the side **a b** has to the homologous side **f g**. Therefore also the polygon **a b c d e** has to the polygon **f g h k l** the duplicate ratio of that which **a b** has to the homologous side **f g**. Wherefore similar polygons, &c. Q. E. D.

COR. 1. In like manner, it may be proved, that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles. Therefore, universally, similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

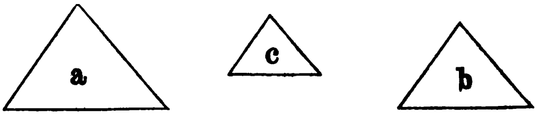
COR. 2. And if to **a b**, **f g**, two of the homologous sides, a third proportional **m** be taken, **a b** has (v. *def.* 10) to **m** the duplicate ratio of that which **a b** has to **f g**; but the four-sided figure or polygon upon **a b** has to the four-sided figure or polygon upon **f g** likewise the duplicate ratio of that which **a b** has to **f g**: therefore, as **a b** is to **m**, so is the figure upon **a b** to the figure upon **f g**, which was also proved in triangles (vi. 19, *cor.*). Therefore, universally, it is manifest that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first to a similar and similarly described rectilineal figure upon the second.

PROPOSITION XXI.—THEOREM.

Rectilineal figures which are similar to the same rectilineal figure are also similar to one another.

LET each of the rectilineal figures **a**, **b** be similar to the rectilineal figure **c**: the figure **a** is similar to the figure **b**.

Because **a** is similar to **c**, they are equiangular, and also have their sides about the equal angles proportionals (vi. *def.* 1). Again, because **b**



is similar to **c**, they are equiangular, and have their sides about the equal angles proportionals (vi. *def.* 1). Therefore the figures **a**, **b** are each of them equiangular to c, and have the sides about the equal angles of each of them and of c proportionals. Wherefore the rectilineal figures a and b are equiangular (i. *ax.* 1), and have their sides about the equal angles proportionals (v. 11). Therefore a is similar (vi. *def.* 1) to b. Q. E. D.

PROPOSITION XXII.—THEOREM.

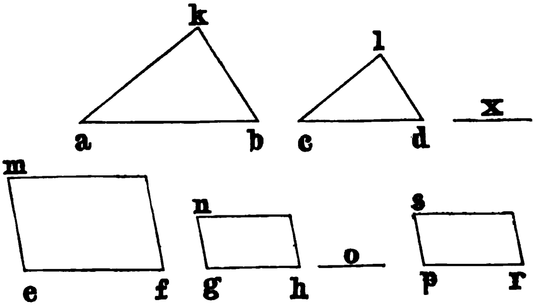
If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

LET the four straight lines a b, c d, e f, g h be proportionals, viz. a b to c d, as e f to g h, and upon a b, c d let the similar rectilineal figures k a b, l c d be similarly described; and upon e f, g h the similar rectilineal figures m f, n h, in like manner. The rectilineal figure k a b is to l c d as m f to n h.

To a b, c d take a third proportional (vi. 11) x; and to e f, g h a third proportional o: and because a b is to c d as e f to g h, and that c d is (v. 11) to x as g h to o; wherefore, *ex æquali* (v. 22), as a b to x, so is e f to o: but as a b to x, so is (vi. 20, *cor.* 2) the rectilineal k a b to the rectilineal l c d, and as e f to o, so is (vi. 20, *cor.* 2) the rectilineal m f to the rectilineal n h. Therefore, as k a b to l c d, so (v. 11) is m f to n h.

And if the rectilineal k a b be to l c d as m f to n h; the straight line a b is to c d as e f to g h.

Make (vi. 12) as a b to c d, so e f to p r, and upon p r describe (vi. 18) the rectilineal figure s r similar and similarly situated to either of the figures m f, n h: then, because as a b to c d, so is e f to p r, and that upon a b, c d are described the similar and similarly situated rectilineals k a b, l c d, and upon e f, p r, in like manner, the similar rectilineals m f, s r; k a b is to l c d, as m f to s r; but, by the hypothesis, k a b is to l c d as m f to n h; and therefore the rectilineal m f having the same ratio to each of the two n h, s r, these are equal (v. 9) to one another: they are also similar and similarly situated; therefore g h is equal to p r. And because as a b to c d, so is e f to p r, and that p r is equal to g h; a b is to c d as e f to g h. If therefore four straight lines, &c. Q. E. D.

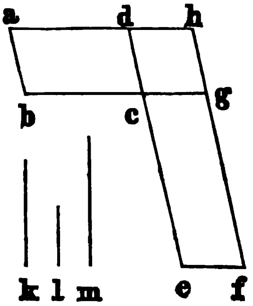


PROPOSITION XXIII.—THEOREM.

Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

LET **a c**, **c f** be equiangular parallelograms, having the angle **b c d** equal to the angle **e c g** : the ratio of the parallelogram **a c** to the parallelogram **c f** is the same with the ratio which is compounded of the ratios of their sides.

Let **b c**, **c g** be placed in a straight line; therefore **d c** and **c e** are also in a straight line (i. 14); and complete the parallelogram **d g**; and taking any straight line **k**, make (vi. 12) as **b c** to **c g**, so **k** to **l**; and as **d c** to **c e**, so make (vi. 12) **l** to **m** : therefore the ratios of **k** to **l**, and **l** to **m**, are the same with the ratios of the sides. viz. of **b c** to **c g**, and **d c** to **c e**. But the ratio of **k** to **m** is that which is said to be compounded (v. *def.* A.) of the ratios of **k** to **l**, and **l** to **m** : wherefore also **k** has to **m** the ratio compounded of the ratios of the sides. And because as **b c** to **c g**, so is the parallelogram **a c** to the parallelogram **c h** (vi. 1); but as **b c** to **c g**, so is **k** to **l**; therefore **k** is (v. 11) to **l**, as the parallelogram **a c** to the parallelogram **c h**. Again, because as **d c** to **c e**, so is the parallelogram **c h** to the parallelogram **c f**; but as **d c** to **c e**, so is **l** to **m**; wherefore **l** is (v. 11) to **m**, as the parallelogram **c h** to the parallelogram **c f** : therefore since it has been proved, that as **k** to **l**, so is the parallelogram **a c** to the parallelogram **c h**; and as **l** to **m**, so the parallelogram **c h** to the parallelogram **c f** : *ex æquali* (v. 22), **k** is to **m**, as the parallelogram **a c** to the parallelogram **c f** : but **k** has to **m** the ratio which is compounded of the ratios of the sides; therefore also the parallelogram **a c** has to the parallelogram **c f** the ratio which is compounded of the ratios of the sides. Wherefore, equiangular parallelograms, &c. Q. E. D.

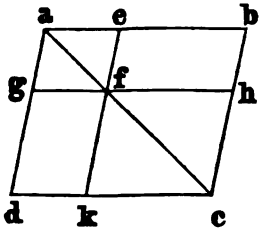


PROPOSITION XXIV.—THEOREM.

The parallelograms about the diameter of any parallelogram are similar to the whole and to one another.

LET **a b c d** be a parallelogram, of which the diameter is **a c**; and **e g**, **h k** the parallelograms about the diameter. The parallelograms **e g**, **h k** are similar both to the whole parallelogram **a b c d**, and to one another.

Because **d c**, **g f** are parallels, the angle **a d c** is equal (i. 29) to the angle **a g f**. For the same reason, because **b c**, **e f** are parallels, the angle **a b c** is equal to the angle **a e f** : and each of the angles **b c d**, **e f g** is equal to the opposite angle **d a b** (i. 34), and therefore are equal to one another : wherefore the parallelograms **a b c d**, **a e f g**, are equiangular : and because the angle **a b c** is equal to the angle **a e f**, and the angle **b a c** common to the two triangles **b a c**, **e a f**, they are equiangular to one another; therefore (vi. 4) as **a b** to **b c**, so is **a e** to **e f**: and because the opposite sides of parallelograms are equal to one another (i. 34), **a b** (v. 7) is to **a d** as **a e** to **a g**; and **d c** to **c b** as **g f** to **f e**; and also **c d** to **d a** as **f g** to **g a**: therefore the sides of the parallelograms **a b c d**, **a e f g** about the equal angles are proportionals; and they are therefore similar to one another (vi. *def.* 1); for the same reason the parallelogram **a b c d** is similar to the parallelogram **f h c k**. Wherefore each of the parallelograms **g e**, **k h** is similar to **d b**: but rectilineal figures which are similar to the same rectilineal figure are also similar to one another (vi. 21); therefore the parallelogram **g e** is similar to **k h**. Wherefore the parallelogram, &c. Q. E. D.

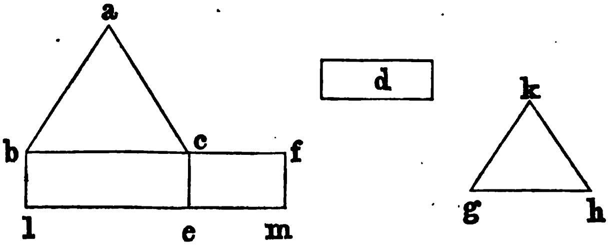


Proposition XXV.—Problem.

To describe a rectilineal figure which shall be similar to one and equal to another given rectilineal figure.

Let **a b c** be the given rectilineal figure to which the figure to be described is required to be similar, and **d** that to which it must be equal. It is required to describe a rectilineal figure similar to **a b c**, and equal to **d**.

Upon the straight line **b c** describe (i. 45, *cor.*) the parallelogram **b e** equal to the figure **a b c**; also upon **c e** describe (i. 45, *cor.*) the parallelogram **c m** equal to **d**, and having the angle **f c e** equal to the angle **c b l**: therefore **b c** and **c f** are in a straight line (i. 29, and i. 14), as also **l e** and **e m**: between **b c** and **c f** find (vi. 13) a mean proportional **g h**, and upon **g h** describe (vi. 18) the rectilineal figure **k g h** similar and similarly situated to the figure **a b c**: and because **b c** is to **g h** as **g h** to **c f**; and if three straight lines be proportionals, as the first is to the third, so is (vi. 20, *cor.* 2) the figure upon the first to the similar and similarly described figure upon the second; therefore, as **b c** to **c f**, so is the rectilineal figure **a b c** to **k g h**: but as **b c** to **c f**, so is (vi. 1) the parallelogram **b e** to the parallelogram **e f**: therefore as the rectilineal figure **a b c** is to **k g h**, so is the parallelogram **b e** to the parallelogram **e f** (v. 11): and the rectilineal figure **a b c** is equal to the parallelogram **b e**; therefore the rec-

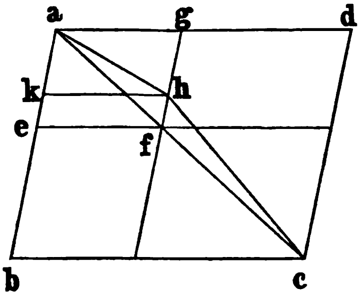


tilineal figure **kgh** is equal (v. 14) to the parallelogram **ef**: but **ef** is equal to the figure **d**; wherefore also **kgh** is equal to **d**; and it is similar to **abc**. Therefore the rectilineal figure **kgh** has been described similar to the figure **abc**, and equal to **d**. Which was to be done.

PROPOSITION XXVI.—THEOREM.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

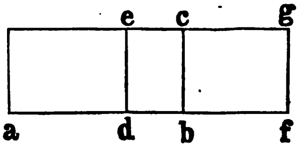
LET the parallelograms **abcd**, **aefg** be similar and similarly situated, and have the angle **dab** common: **abcd** and **aefg** are about the same diameter.



For, if not, let, if possible, the parallelogram **bd** have its diameter **ahc** in a different straight line from **af**, the diameter of the parallelogram **eg**, and let **gf** meet **ahc** in **h**; and through **h** draw **hk** parallel to **ad** or **bc**: therefore the parallelograms **abcd**, **akhg** being about the same diameter, they are similar to one another (vi. 24): wherefore as **da** to **ab**, so is (vi. *def.* 1) **ga** to **ak**: but because **abcd** and **aefg** are similar parallelograms, as **da** is to **ab**, so is **ga** to **ae**; therefore (v. 11) as **ga** to **ae**, so is **ga** to **ak**; wherefore **ga** has the same ratio to each of the straight lines **ae**, **ak**; and consequently **ak** is equal (v. 9) to **ae**, the less to the greater, which is impossible: therefore **abcd** and **akhg** are not about the same diameter: wherefore **abcd** and **aefg** must be about the same diameter. Therefore, if two similar parallelograms, &c. Q. E. D.

To understand the three following propositions more easily, it is to be observed,

1. That a parallelogram is said to be applied to a straight line, when it is described upon it as one of its sides. For example, the parallelogram **ac** is said to be applied to the straight line **ab**.



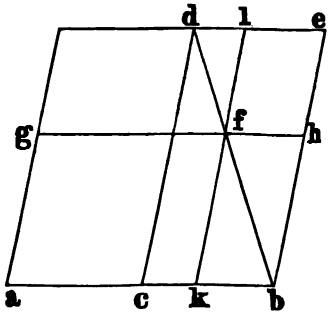
2. But a parallelogram **ae** is said to be applied to a straight line **ab**, deficient by a parallelogram, when **ad** the base of **ae** is less than **ab**, and therefore **ae** is less than the parallelogram **ac** described upon **ab** in the same angle, and between the same parallels, by the parallelogram **dc**: and **dc** is therefore called the defect of **ae**.

3. And a parallelogram **ag** is said to be applied to a straight line **ab**, exceeding by a parallelogram, when **af** the base of **ag** is greater than **ab**, and therefore **ag** exceeds **ac** the parallelogram described upon **ab** in the same angle, and between the same parallels, by the parallelogram **bg**.

PROPOSITION XXVII.—THEOREM.

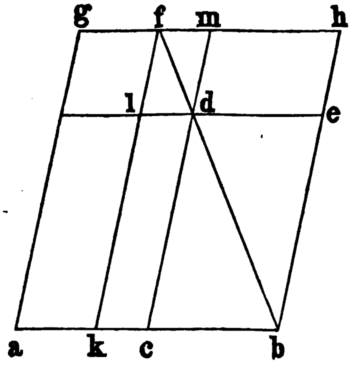
Of all parallelograms applied to the same straight line, and deficient by parallelograms similar and similarly situated to that which is described upon the half of the line, that which is applied to the half, and is similar to its defect, is the greatest.

LET **a b** be a straight line divided into two equal parts in **c**, and let the parallelogram **a d** be applied to the half **a c**, which is therefore deficient from the parallelogram upon the whole line **a b** by the parallelogram **c e** upon the other half **c b**; of all the parallelograms applied to any other parts of **a b**, and deficient by parallelograms that are similar and similarly situated to **c e**, **a d** is the greatest.



Let **a f** be any parallelogram applied to **a k**, any other part of **a b** than the half, so as to be deficient from the parallelogram upon the whole line **a b** by the parallelogram **k h** similar and similarly situated to **c e**: **a d** is greater than **a f**.

First, let **a k** the base of **a f** be greater than **a c** the half of **a b**; and because **c e** is similar to the parallelogram **k h**, they are about the same diameter (vi. 26): draw their diameter **d b**, and complete the scheme: because the parallelogram **c f** is equal (i. 43) to **f e**, add **k h** to both, therefore the whole **c h** is equal to the whole **k e**: but **c h** is equal (i. 36) to **c g**, because the base **a c** is equal to the base **c b**; therefore **c g** is equal to **k e**: to each of these add **c f**; then the whole **a f** is equal to the gnomon **c h l**, therefore **c e**, or the parallelogram **a d**, is greater than the parallelogram **a f**.

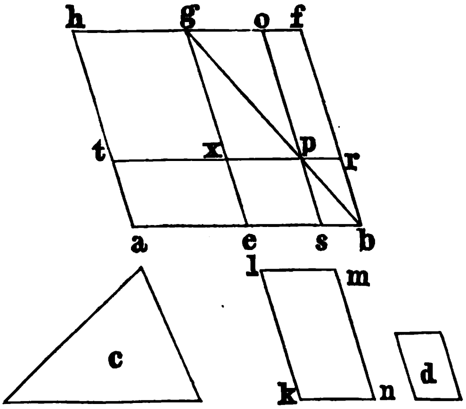


Next, let **a k** the base of **a f** be less than **a c**, and the same construction being made, the parallelogram **d h** is equal to **d g** (i. 36), for **h m** is equal to **m g** (i. 34), because **b c** is equal to **c a**; wherefore **d h** is greater than **l g**: but **d h** is equal (i. 43) to **d k**; therefore **d k** is greater than **l g**; to each of these add **a l**; then the whole **a d** is greater than the whole **a f**. Therefore, of all parallelograms applied, &c. Q. E. D.

PROPOSITION XXVIII.—PROBLEM.

To a given straight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram; but the given rectilineal figure to which the parallelogram to be applied is to be equal must not be greater than the parallelogram applied to half of the given line, having its defects similar to the defect of that which is to be applied, that is, to the given parallelogram.

LET **a b** be the given straight line, and **c** the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure must not be greater than the parallelogram applied to the half of the line, having its defect from that upon the whole line similar to the defect of that which is to be applied; and let **d** be the parallelogram to which this defect is required to be similar. It is required to apply a parallelogram to the straight line **a b**, which shall be equal to the figure **c**, and be deficient from the parallelogram upon the whole line by a parallelogram similar to **d**.



Divide **a b** into two equal parts (i. 10) in the point **e**, and upon **e b** describe the parallelogram **e b f g** similar (vi. 18) and similarly situated to **d**, and complete the parallelogram **a g**, which must either be equal to **c**, or greater than it, by the determination: and if **a g** be equal to **c**, then what was required is already done: for, upon the straight line **a b**, the parallelogram **a g** is applied equal to the figure **c**, and deficient by the parallelogram **e f** similar to **d**: but, if **a g** be not equal to **c**, it is greater than it; and **e f** is equal to **a g**; therefore **e f** also is greater than **c**. Make (vi. 25) the parallelogram **k l m n** equal to the excess, of **e f** above **c**, and similar and similarly situated to **d**; but **d** is similar to **e f**, therefore (vi. 21) also **k m** is similar to **e f**. Let **k l** be the homologous side to **e g**, and **l m** to **g f**; and because **e f** is equal to **c** and **k m** together, **e f** is greater than **k m**; therefore the straight line **e g** is greater than **k l**, and **g f** than **l m**: make **g x** equal to **l k**, and **g o** equal to **l m**, and complete the parallelogram **x g o p**: therefore **x o** is equal and similar to **k m**; but **k m** is similar to **e f**; wherefore also **x o** is similar to **e f**, and therefore **x o** and **e f** are about the same diameter (vi. 26): let **g p b** be their diameter, and complete the scheme. Then, because **e f** is equal to **c** and **k m** together, and **x o** a part of the one is equal to **k m** a part of the other, the remainder, viz. the gnomon **e r o** is equal to the remainder **c**: and because **o r** is equal (i. 34) to **x s**, by adding **s r** to each, the whole **o b** is equal to the whole **x b**: but **x b** is equal (i. 36) to **t e**, because the base

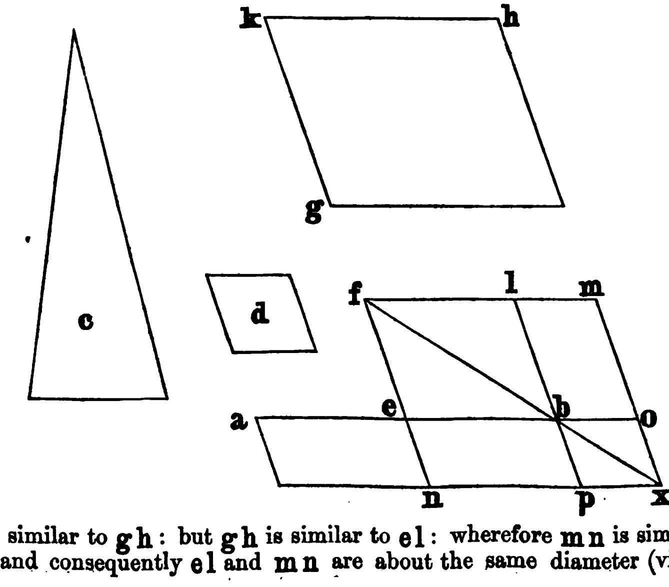
ae is equal to the base **eb**; wherefore also **te** is equal to **ob**; add **xs** to each, then the whole **ts** is equal to the whole, viz. to the gnomon **ero**: but it has been proved that the gnomon **ero** is equal to **c**, and therefore also **ts** is equal to **c**. Wherefore the parallelogram **ts**, equal to the given rectilineal figure **c**, is applied to the given straight line **ab** deficient by the parallelogram **sr**, similar to the given one **d**, because **sr** is similar to **ef** (vi. 24). Which was to be done.

PROPOSITION XXIX.—PROBLEM.

To a given straight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given.

Let **ab** be the given straight line, and **c** the given rectilineal figure to which the parallelogram to be applied is required to be equal, and **d** the parallelogram to which the excess of the one to be applied above that upon the given line is required to be similar. It is required to apply a parallelogram to the given straight line **ab**, which shall be equal to the figure **c**, exceeding by a parallelogram similar to **d**.

Divide **ab** into two equal parts in the point **e**, and upon **eb** describe (vi. 18) the parallelogram **el** similar and similarly situated to **d**: and make (vi. 25) the parallelogram **gh** equal to **el** and **c** together, and similar and similarly situated to **d**; wherefore **gh** is similar to **el** (vi. 21): let **kh** be the side homologous to **fl**, and **kg** to **fe**: and because the parallelogram **gh** is greater than **el**, therefore the side **kh** is greater than **fl**, and **kg** than **fe**. Produce **fl** and **fe**, and make **flm** equal to **kh**, and **fen** to **kg**, and complete the parallelogram **mn**: **mn** is therefore equal and similar to **gh**: but **gh** is similar to **el**: wherefore **mn** is similar to **el**, and consequently **el** and **mn** are about the same diameter (vi. 26);

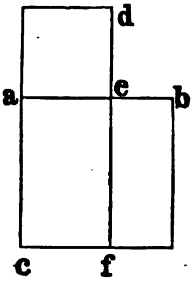


draw their diameter **fx**, and complete the scheme. Therefore since **gh** is equal to **el** and **c** together, and that **gh** is equal to **mn**; **mn** is equal to **el** and **c**: take away the common part **el**; then the remainder, viz. the gnomon **nol**, is equal to **c**. And because **ae** is equal to **eb**, the parallelogram **an** is equal (i. 36) to the parallelogram **nb**, that is, to **bm** (i. 43). Add **no** to each; therefore the whole, viz. the parallelogram **ax**, is equal to the gnomon **nol**. But the gnomon **nol** is equal to **c**; therefore also **ax** is equal to **c**. Wherefore to the straight line **ab** there is applied the parallelogram **ax** equal to the given rectilineal **c**, exceeding by the parallelogram **po**, which is similar to **d**, because **po** is similar to **el** (vi. 24). Which was to be done.

PROPOSITION XXX.—PROBLEM.

To cut a given straight line in extreme and mean ratio.

LET **ab** be the given straight line, it is required to cut it in extreme and mean ratio.



Upon **ab** describe (i. 46) the square **bc**, and to **ac** apply the parallelogram **cd**, equal to **bc**, exceeding by the figure **ad** similar to **bc** (vi. 29): but **bc** is a square, therefore also **ad** is a square; and because **bc** is equal to **cd**, by taking the common part **ce** from each, the remainder **bf** is equal to the remainder **ad**; and these figures are equiangular, therefore their sides about the equal angles are reciprocally proportional (vi. 14): wherefore, as **fe** to **ed**, so is **ae** to **eb**: but **fe** is equal to **ac** (i. 34), that is, to **ab**; and **ed** is equal to **ae**: therefore as **ba** to **ae**, so is **ae** to **eb**: but **ab** is greater than **ae**; wherefore **ae** is greater than **eb** (v. 14). Therefore the straight line **ab** is cut in extreme and mean ratio in **e** (vi. *def.* 3). Which was to be done.

Otherwise,

Let **ab** be the given straight line; it is required to cut it in extreme and mean ratio.

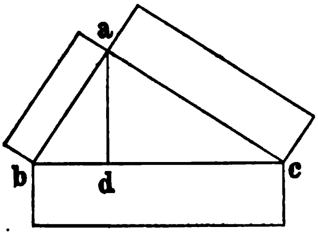
Divide **ab** in the point **c**, so that the rectangle contained by **ab**, **bc** be equal to the square of **ac** (ii. 11): then, because the rectangle **ab**, **bc**, is equal to the square of **ac**; as **ba** to **ac**, so is **ac** to **cb** (vi. 17): therefore **ab** is cut in extreme and mean ratio in **c** (vi. *def.* 3). Which was to be done.

PROPOSITION XXXI.—THEOREM.

In right-angled triangles the rectilineal figure described upon the side opposite to the right angle is equal to the similar and similarly described figures upon the sides containing the right angle.

LET **abc** be a right-angled triangle, having the right angle **bac**. The rectilineal figure described upon **bc** is equal to the similar and similarly described figures upon **ba**, **ac**.

Draw the perpendicular **ad**; therefore, because in the right-angled triangle **abc**, **ad** is drawn from the right angle at **a** perpendicular to the base **bc**, the triangles **abd**, **adc** are similar to the whole triangle **abc**, and to one another (vi. 8), and because the triangle **abc** is similar to **abd**; as **cb** to **ba**, so is **ba** to **bd** (vi. 4); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the first to the similar and similarly described figure upon the second (vi. 20, *cor.* 2): therefore as **cb** to **bd**, so is the figure upon **cb** to the similar and similarly described figure upon **ba**: and inversely (v. B.). as **db** to **bc**, so is the figure upon **ba** to that upon **bc**. For the same reason, as **dc** to **cb**, so is the figure upon **ca** to that upon **cb**. Wherefore as **bd** and **dc** together to **bc**, so are the figures upon **ba**, **ac** to that upon **bc** (v. 24): but **bd** and **dc** together are equal to **bc**. Therefore the figure described on **bc** is equal (v. A.) to the similar and similarly described figures on **ba**, **ac**. Wherefore, in right-angled triangles, &c. Q. E. D.

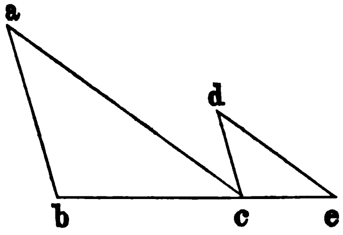


Proposition XXXII.—Theorem.

If two triangles which have two sides of the one proportional to two sides of the other be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

Let **abc**, **dce** be two triangles which have the two sides **ba**, **ac** proportional to the two **cd**, **de**, viz. **ba** to **ac** as **cd** to **de**; and let **ab** be parallel to **dc**, and **ac** to **de**, **bc** and **ce** are in a straight line.

Because **ab** is parallel to **dc**, and the straight line **ac** meets them, the alternate angles **bac**, **acd** are equal (i. 29); for the same reason, the angle **cde** is equal to the angle **acd**; wherefore also **bac** is equal to **cde**: and because the triangles **abc**, **dce** have one angle at **a** equal to one at **d**, and the sides about these angles proportionals, viz. **ba** to **ac** as **cd** to **de**, the triangle **abc** is equiangular (vi. 6) to **dce**: therefore the angle **abc** is equal to the angle **dce**: and the angle **bac** was proved to be equal to **acd**: therefore the whole angle **ace** is equal to the two angles **abc**, **bac**; add the common angle **acb**, then the angles **ace**, **acb** are equal to the angles **abc**, **bac**, **acb**: but **abc**, **bac**, **acb** are equal to two right angles (i. 32); therefore also the angles **ace**, **acb** are equal to two right angles. And since at the point **c**, in the



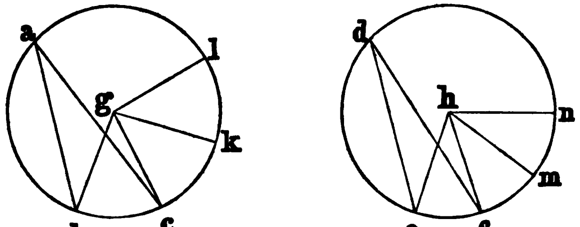
straight line **a c**, the two straight lines **b c**, **c e**, which are on the opposite sides of it, make the adjacent angles **a c e**, **a c b** equal to two right angles; therefore (i. 14) **b c** and **c e** are in a straight line. Wherefore, if two triangles, &c. Q. E. D.

PROPOSITION XXXIII.—THEOREM.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the circumferences on which they stand have to one another; so also have the sectors.

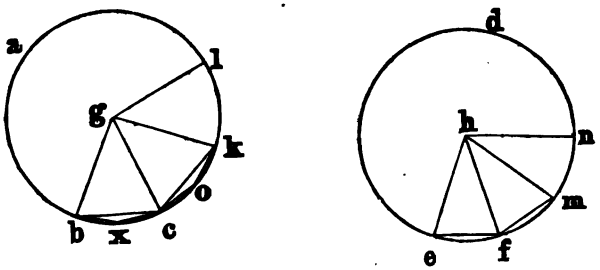
LET **a b c**, **d e f** be equal circles; and at their centres the angles **b g c**, **e h f**, and the angles **b a c**, **e d f**, at their circumferences; as the circumference **b c** to the circumference **e f**, so is the angle **b g c** to the angle **e h f**, and the angle **b a c** to the angle **e d f**: and also the sector **b g c** to the sector **e h f**.

Take any number of circumferences **c k**, **k l**, each equal to **b c**, and any number whatever **f m**, **m n**, each equal to **e f**: and join **g k**, **g l**, **h m**, **h n**. Because the circumferences **b c**, **c k**, **k l** are all equal, the angles **b g c**, **c g k**, **k g l** are also all equal (iii. 27): therefore what multiple soever the circumference **b l** is of the circumference **b c**, the same multiple is the angle **b g l** of the angle **b g c**: for the same reason, whatever multiple the circumference **e n** is of the circumference **e f**, the same multiple is the angle **e h n** of the angle **e h f**: and if the circumference **b l** be equal to the circumference **e n**, the angle **b g l** is also equal (iii. 27) to the angle **e h n**; and if the circumference **b l** be greater than **e n**, likewise the angle **b g l** is greater than **e h n**: and if less, less: there being then four magnitudes, the two circumferences **b c**, **e f**, and the two angles **b g c**, **e h f**; of the circumference **b c**, and of the angle **b g c**, have been taken any equimultiples whatever, viz. the circumference **b l**, and the angle **b g l**; and of the circumference **e f**, and of the angle **e h f**, any equimultiples whatever, viz. the circumference **e n**, and the angle **e h n**: and it has been proved, that if the circumference **b l** be greater than **e n**, the angle **b g l** is greater than **e h n**; and if equal, equal; and if less, less: as therefore the circumference **b c** to the circumference **e f**, so (v. *def.* 5) is the angle **b g c** to the angle **e h f**: but as the angle **b g c** is to the angle **e h f**, so is (v. 15) the angle **b a c** to the angle **e d f**; for each is double of each (iii. 20); therefore, as the circumference **b c** is to **e f**,



so is the angle **bgc** to the angle **ehf**, and the angle **bac** to the angle **edf**.

Also, as the circumference **bc** to **ef**, so is the sector **bgc** to the sector **ehf**. Join **bc**, **ck**, and in the circumferences **bc**, **ck** take any points **x**, **o**, and join **bx**, **xc**, **co**, **ok**: then, because in the triangles **gbc**, **gck** the two sides **bg**, **gc**, are equal to the two **cg**, **gk**, and that they contain equal angles; the base **bc** is equal (i. 4) to the base **ck**, and the triangle **gbc** to the triangle **gck**: and because the circumference **bc** is equal to the circumference **ck**, the remaining part of the whole circumference of the circle **abc**, is equal to the remaining part of the whole circumference of the same circle: wherefore the angle **bxc** is equal to the angle **cok** (iii. 27) and the segment **bxc** is therefore similar to the segment **cok** (iii. *def.* 11); and they are upon equal straight lines **bc**, **ck**: but similar segments of circles upon equal straight lines are equal (iii. 24) to one another: therefore the segment **bxc** is equal to the segment **cok**: and the triangle **bgc** is equal to the triangle **cgk**: therefore the whole, the sector **bgc** is equal to the whole, the sector **cgk**: for the same reason, the sector **kgl** is equal to each of the sectors **bgc**, **cgk**: in the same manner, the sectors **ehf**, **fhm**, **mhn** may be proved equal to one another: therefore, what multiple soever the circumference **bl** is of the circumference **bc**, the same multiple is the sector **bgl** of the sector **bgc**: for the same reason, whatever multiple the circumference **en** is of **ef**, the same multiple is the sector **ehn** of the sector **ehf**; and if the circumference **bl** be equal to **en**, the sector **bgl** is equal to the sector **ehn**; and if the circumference **bl** be greater than **en**, the sector **bgl** is greater than the sector **ehn**; and if less, less: since, then, there are four magnitudes, the two circumferences **bc**, **ef**, and the two sectors **bgc**, **ehf**, and of the circumference **bc**, and sector **bgc**, the circumference **bl** and sector **bgl** are any equal multiples whatever; and of the circumference **ef**, and sector **ehf**, the circumference **en**, and sector **ehn**, are any equimultiples whatever: and that it has been proved, if the circumference **bl** be greater than **en**, the sector **bgl** is greater than the sector **ehn**; and if equal, equal; and if less, less; therefore (v. *def.* 5), as the circumference **bc** is to the circumference **ef**, so is the sector **bgc** to the sector **ehf**. Wherefore, in equal circles, &c. Q. E. D.

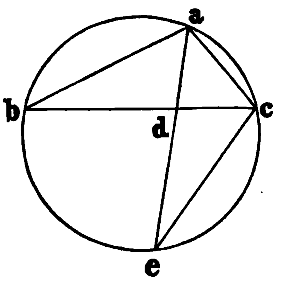


PROPOSITION B.—THEOREM.

If an angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

LET **a b c** be a triangle, and let the angle **b a c** be bisected by the straight line **a d**; the rectangle **b a, a c** is equal to the rectangle **b d, d c**, together with the square of **a d**.

Describe the circle (iv. 5) **a c b** about the triangle, and produce **a d** to the circumference in **e**, and join **e c**. Then because the angle **b a d** is equal to the angle **c a e**, and the angle **a b d** to the angle (iii. 21) **a e c**, for they are in the same segment; the triangles **a b d**, **a e c**, are equiangular to one another: therefore as **b a** to **a d**, so is (vi. 4) **e a** to **a c**, and consequently the rectangle **b a, a c** is equal (vi. 16) to the rectangle **e a, a d** that is (ii. 3), to the rectangle **e d, d a**, together with the square of **a d**: but the rectangle **e d, d a** is equal to the rectangle (iii. 35) **b d, d c**. Therefore the rectangle **b a, a c** is equal to the rectangle **b d, d c**, together with the square of **a d**. Wherefore, if an angle, &c. Q. E. D.

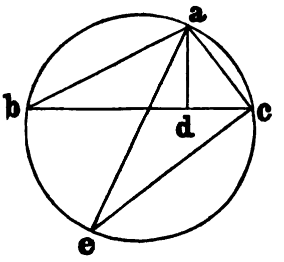


PROPOSITION C.—THEOREM.

If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

LET **a b c** be a triangle, and **a d** the perpendicular from the angle **a** to the base **b c**; the rectangle **b a, a c** is equal to the rectangle contained by **a d**, and the diameter of the circle described about the triangle.

Describe (iv. 5) the circle **a c b** about the triangle, and draw its diameter **a e**, and join **e c**: because the right angle **b d a** is equal (iii. 31) to the angle **e c a** in a semicircle, and the angle **a b d** to the angle **a e c** in the same segment (iii. 21); the triangles **a b d**, **a e c** are equiangular: therefore as (vi. 4) **b a** to **a d**, so is **e a** to **a c**; and consequently the rectangle **b a, a c** is equal (vi. 16) to the rectangle **e a, a d**. If therefore from an angle, &c. Q. E. D.

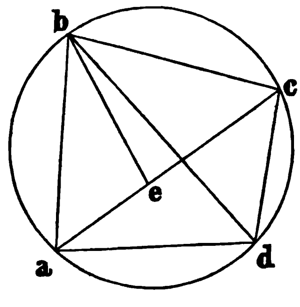


PROPOSITION D.—THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to both the rectangles contained by its opposite sides.

LET **a b c d** be any quadrilateral inscribed in a circle, and join **a c**, **b d**; the rectangle contained by **a c**, **b d** is equal to the two rectangles contained by **a b**, **c d**, and by **a d**, **b c**.*

Make the angle **a b e** equal to the angle **d b c**: add to each of these the common angle **e b d**, then the angle **a b d** is equal to the angle **e b c**: and the angle **b d a** is equal (iii. 21) to the angle **b c e**, because they are in the same segment: therefore the triangle **a b d** is equiangular to the triangle **b c e**: wherefore (vi. 4), as **b c** is to **c e**, so is **b d** to **d a**; and consequently the rectangle **b c**, **a d** is equal (vi. 16) to the rectangle **b d**, **c e**: again, because the angle **a b e** is equal to the angle **d b c**, and the angle (iii. 21) **b a e** to the angle **b d c**, the triangle **a b e** is equiangular to the triangle **b c d**: as therefore **b a** to **a e**, so is **b d** to **d c**; wherefore the rectangle **b a**, **d c** is equal to the rectangle **b d**, **a e**: but the rectangle **b c**, **a d** has been shewn equal to the rectangle **b d**, **c e**; therefore the whole rectangle **a c**, **b d** (ii. 1) is equal to the rectangle **a b**, **d c**, together with the rectangle **a d**, **b c**. Therefore, the rectangle, &c. Q. E. D.



EXERCISES ON BOOK VI.

SECT. I.—PROBLEMS.

1. GIVEN the base, the vertical angle, and the ratio between the two sides, to construct the triangle.

2. Given the vertical angle, the perpendicular from it to the base, and the ratio between the segments and the base on each side of the perpendicular, to construct the triangle.

3. Given a scalene triangle, to construct an isosceles triangle equal to it, and with an equal vertical angle.

4. Given a finite straight line, to describe on it a right-angled triangle whose sides shall be continual proportionals.

5. Given an angle and a point, to draw from the point a straight line cutting the lines that contain the angle, in such a way that the parts be-

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* This is a Lemma of Cl. Ptolemæus, in page 9 of his μεγαλη συνταξις.

tween the vertex and the points of intersection shall have to each other a given ratio.

6. Given a straight line, to divide it into two parts such that the square of one of the parts shall have a given ratio to the rectangle contained by the whole line and the other part.

7. Given the first and fourth of four proportional lines and the difference between the second and third, to find the second and third.

8. Given a rectilineal figure, to describe a square which shall have a given ratio to it.

9. Given a trapezium in which the sides about one angle are equal, and the sides about the opposite angle also equal, to inscribe in it a square.

10. Given two similar rectilineal figures, to find a third similar rectilineal figure which shall be a mean proportional between them.

11. Given a segment of a circle, to divide it into two parts so that the chords of these parts shall have a given ratio to each other.

12. To trisect a given circle.

SECT. II.—THEOREMS.

13. If a straight line be drawn from the vertex of an isosceles triangle at right angles to one of the equal sides and produced until it meets the base produced, either of the equal sides is a mean proportional between the base and one-half of the base produced.

14. If the three sides of a triangle be bisected and straight lines drawn from the points of bisection to the opposite angles, they shall intersect each other in the same point.

15. If two or more lines meet three parallel lines, they are cut proportionally.

16. If a straight line touch two circles that also touch each other, the mean proportional between the diameters of the circles is that part of the line lying between the points of contact.

17. If parallelograms be equiangular, the ratio between them is the same as the ratio between the rectangles contained by the sides about equal angles in each.

THE END.

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"The very handsome volume before us, a beautifully-printed octavo of nearly 1500 pages, is a condensation of the two-volume quarto work of Dr. Webster. The single-volume version is taken from the last edition of the original work, with all its improvements and additions. The plan of the abridgment is excellent; it contains the leading and most important etymologies as given in the quarto edition. The definitions remain unaltered, except by an occasional compression in their statement. All the significations of words, also, as exhibited in the larger work, remain unaltered; but the illustrations and authorities are only inserted in doubtful and disputed cases. It will be seen by this, that the smaller edition has been produced mainly by the very common-sense process of omitting the examples in the cases of undisputed and ordinary, or, at all events, not uncommon words, the authenticity and correctness of the definition of which nobody ever thinks of doubting. The richness of the treasury of words given is immense. It includes every possible scientific and artistic term; and the Doctor has been very liberal in inserting all those expressions adopted from the French, and commonly used in the upper circles of this country, as well as in recognising the semi-slangish expressions of familiarly colloquial life. The definitions are very clear, and quite devoid of Johnson's occasional pedantry; as, for example, let the reader 'turn up' the definition of network in both the dictionaries. The number of new words not to be found in other works of the same nature is immense; and in the preface it is asserted, and with truth, that thousands more could very easily have been added. The difficulty would appear to have been the selection. Thousands of words, it is remarked, can be found in dictionaries which have, indeed, been proposed, but never adopted in the language. These, as a general rule, Dr. Webster excludes; but he is careful to preserve those terms, once household words, and which have now fallen into disuse, but which were yet employed by the old standard writers, and a knowledge of which is necessary to understand some of the finest phases of our literature. A good many of the new words introduced are marked 'familiar,' 'colloquial,' or 'low.' The dictionary is rich in synonyms—a very good feature—and one on which great care has been bestowed. . . . To all who wish for the most complete, cheap, and portable dictionary at this moment

existing of our noble language, including an immense mass of philologic matter—copious vocabularies of mythologic, Scriptural, and geographic names—some curious dissertation as to the tendencies of orthography, and on the elements of the pronunciation of the principal continental languages—we can cordially recommend the beautifully-printed and elaborately got-up volume before us."—ATLAS.

"Professor Webster's English Dictionary has long become a standard English work, albeit the compiler was an American. The present volume is a revised edition of that great work—for such it strictly is—published in a cheap form. Originally given to the world in a quarto form, it was found before long, that an edition in a more compact shape was loudly called for; and accordingly, in the year 1829, Professor Worcester, of Cambridge, Massachusets, undertook its abbreviation and reduction, under the direction of Dr. Webster himself. The additions and alterations of the larger work (ed. 1840) were subsequently incorporated with the abbreviated edition; and this edition is the basis of the one now published by the proprietors of the 'Illustrated London Library.' It is consequently made, in all respects, consistent with the larger work; and it presents, accordingly, on a reduced scale, a clear, accurate, and full exhibition of the American dictionary in its every part.

"The meaning of every English word, in all its various shades, is given in this admirable work, which for completeness far excels the dictionary of Dr. Johnson; and it contains, moreover, a dictionary of synonyms, somewhat on the foundation of Carpenter's small publication, but more elaborately wrought and more fully treated. Under each of the important words, all others having the same general signification are arranged together, except in cases where they have been previously exhausted in framing the definitions—a system which is of the utmost importance to young writers desirous of attaining grace, variety, and copiousness of diction.

"Some thousands of new words have been added to this vocabulary; but in justice to the compiler it should be stated that many of them are obsolete terms, employed by the writers of the Elizabethan period, but since their day fallen completely into disuse. The great bulk of these additions, however, consists of familiar and colloquial words and technical expressions necessitated by the establishment of novel occupations and new sources of ideas. The *Americanisms,* as such, are not so numerous as might be supposed; and the great bulk of them retained are shown to be old words with new meanings. All the scientific terms have been subjected to the strictest supervision on the part of competent persons; and so likewise have those of an artistic tendency. The result is such a dictionary as the English language has never had before; and which is as much superior to the last edition of Johnson's great work, even with all its improvements, as that stupendous production is to the moderate octavo of Bailey."—OBSERVER.

"The advantage to the literary public of a cheap, enlarged, improved, and beautifully printed edition of a dictionary of our language, which has attained a considerable reputation, will be highly appreciated, and will serve very materially to assist its critical study. By dint of immense labour, continued during the space of three years, the editor of this fine volume was able to augment considerably the value of the original work, and in 1847 gave to the world the result of his extensive collations and emendations. Amongst the more remarkable features of this revised Dictionary may be mentioned that of its being a Dictionary of synonyms, though not one in which, like Crabb's, nice shades of difference in the meaning of words are elaborately investigated. On the contrary, the practical force of each word is simply and accurately stated in such a manner as to leave no doubt or difficulty in selection. Great labour has been devoted to the perfection of the definitions, a subject in respect of which Webster's Dictionary from the first enjoyed a high reputation. In every branch of science, literature, art, and commercial economy, the best authors have been had recourse to, and their definitions extracted wherever necessary. In the course of the revision of the work some thousands of new words have been added, important alterations made in the orthography of peculiar words, to keep pace with the tendencies of our mode of writing, and particular care taken to exhibit, as perfectly as possible, the true pronunciation. Some supplementary vocabularies of classical names, and strictures on the right pronunciation of classical terms and names, complete the book, which, upon the whole, will be found a most valuable addition to the library of the most zealous reader. It contains 1300 large and rather closely-printed pages, and the typography is remarkably fine."—MORNING ADVERTISER.

"It is somewhat strange that we should be indebted to American authorship for a Dictionary of the English language. A careful examination of the contents, however, would induce us to hail the work as a literary contribution of great value, and we do not

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know any subject to which an American could devote his time and talents more appropriately; for the time is not far distant when the English language will be spoken on the other side of the Atlantic by a vastly greater number of human beings than at present inhabit the parent isles. The volume before us is an abridgment of Dr. Webster's great work, which has been well and favourably known both here and in America for many years; but abridged though it be, the book now issued by the enterprising London publishers extends to about 1500 closely printed large octavo pages. The abridgment has been made on the plan of leaving out only the illustrations and authorities; but for everyday use the work before us will serve almost every purpose; and at the same time a degree of cheapness has been secured which will place such a necessary and comprehensive book in the hands of thousands, who have been hitherto obliged to content themselves with cheap, and often indifferent, small editions of Johnson, Walker, and other eminent lexicographers. It appears that Professor Goodrich, assisted by eminent men in every department of literature and science on both sides of the Atlantic, has been engaged on the present work for three years, and the result is perhaps the most complete dictionary extant upon the same scale. Though the work is of American manufacture, the authorities followed by Dr. Webster and his editor are, in almost every instance, the standard English authors, from Shakspeare downwards. Many thousands of new words, which have grown up with the wants and accidents of society, have been included. Indeed, we believe all of this modern class which deserve to be retained are here included, and in a general way the definitions will be found unimpeachable."—GLASGOW HERALD.

"A new edition of Webster's well-known and highly esteemed dictionary. It is the most complete work of the kind yet published. The definitions are clear and concise, presenting briefly the various meanings and shades of meaning belonging to each word. It contains an immense number of words, connected with every department of literature, art, and science, not to be found in the ordinary dictionaries. The present editor, Professor Goodrich, has added, he states, some thousands in the course of revision. The pronunciation is satisfactorily indicated, and in most cases the synonyms of the words defined are added—a great advantage to persons engaged in literary composition. Further, the work includes a synopsis of words differently pronounced by different orthoepists, Walker's key to the classical pronunciation of Greek, Latin, and Scripture proper names, and a vocabulary of modern geographical names, with their pronunciation."—LEEDS TIMES.

"Webster's Dictionary is the production of the greatest lexicographer that has ever lived. He combined in the fullest measure the two great requisites, the highest literary talent and the most extensive attainments in philology. The present edition is the abridgment which was made by Mr. Worcester under Dr. Webster's direction in 1829. It only omits the passages cited to illustrate the definitions; and these, we think, are of very little consequence either one way or the other, as the lexicographer's authority should be sufficient. We believe it is one of the cheapest and best editions of the work that have been published in this country."—ENGLISH JOURNAL OF EDUCATION.

"A marvel of accuracy, neatness, and cheapness. A volume which gives the origin, correct spelling and pronunciation, and a definition or explanation of more than one hundred thousand words must be a wonderful book. It contains 27,000 more words than the most modern edition of Johnson's great dictionary. A friend on reading that Dr. Adam Clarke had been employed during a quarter of a century on his invaluable Commentary, immediately sent for a copy. Dr. Webster spent a large portion of fourscore years in compiling, correcting, and perfecting this Dictionary. It displays extensive research, patient investigation, carefulness in the smallest details, combined with a sound judgment, a clear head, and untiring energy. It is a contribution of substantial service not only to our times, but for posterity. The literary editorship of this new edition has been wisely entrusted to, and as wisely completed by Professor Goodrich, of Yale College. All the ancient and modern scientific words have been submitted to learned professors in the various branches to which they belong. We cannot specify all the addenda prefixed and affixed, but a copious geographical vocabulary is a valuable and useful feature, following Scripture proper names. We have been delighted with Dr. Webster's Seventeen Rules for the Corrected Orthography of the English Language. By their adoption some thousands of common errors will for the future be avoided, and a uniform mode of writing the same class of words will be the beneficial result."—WESLEY BANNER.

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